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THE HISTORY OF CALCULUS*

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Everyone knows that Newton and Leibniz are the founders of Calculus. Some may think it suffices to know just this one fact. But it is worthwhile, indeed, to go into more details and to study the history of the development of Calculus, in particular, up to the time of Newton and Leibniz.

In our courses on Calculus we usually begin with differentiation and then come later to integration. This is entirely justified, since differentiation is simpler and easier than integration. On the other hand, the historical development starts with integration; computing areas, volumes, or lengths of arcs were the first problems occurring in the history of Calculus. Such problems were discussed by ancient Greek mathematicians, especially by Archimedes, whose outstanding and penetrating achievements mark the peak of all ancient mathematics and also the very beginning of the theory of integration. The method applied by Archimedes for his proofs was the so-called method of exhaustion, that is, in the case of plane areas, the method of inscribed and circumscribed polygons with an increasing number of edges. This method was first rigorously applied, in the form of a double *reductio ad absurdum*, by the great Greek mathematician Eudoxus at the beginning of the fourth century B.C. He first proved the facts, previously stated by Democritus, that the volume of a pyramid equals one third of the corresponding prism and the volume of a cone equals one third of the corresponding cylinder. The same method was also used by Euclid and then with the greatest success by Archimedes (third century B.C.). It is well known that Archimedes was the first to determine the area and the length of the circle, that is, to give suitable approximate values of π, and moreover to determine the volume and the area of the surface of the sphere and of cylinders and cones. But he went far beyond this [1]; he found the areas of ellipses, of parabolic segments, and also of sectors of a spiral, the volumes of segments of the solids of revolution of the second degree, the centroids of segments of a parabola, of a cone, of a segment of the sphere, of right segments of a paraboloid of revolution and of a spheroid. These were amazing achievements, indeed. Archimedes proved his results in the classical manner, by the method of exhaustion. Sometimes the type of approximation is just the same as we would use. For instance, in order to obtain the volume of a solid of revolution of the second degree, Archimedes approximates the volume by a sum of cylindrical slabs. But the direct evaluation of the limit of such sums was cumbersome. Hence we may ask: what was the method used by Archimedes for finding his results?

There is an indication of his method in the beginning of his book on the quadrature of the parabola. But a full explanation of his procedure was given by him in a work rediscovered as late as 1906. It is his *Method Concerning Mechanical Theorems, dedicated to Eratosthenes*, known as Archimedes' *Method*

* Based on two addresses given at the Mathematics Club of Purdue University, June 2, 1949, and at the meeting of the Indiana Academy of Science, Crawfordsville, Ind., November 4, 1949.

or ἔφοδος [2]. This manuscript was found in Istanbul as a so-called palimpsest. That is to say, in the 10th century A.D. the manuscript of the *Method* was written on this parchment; later, in the 13th century, since nobody there was interested in it any longer or could even understand it, the *Method* was washed off and a religious text of the orthodox church, a so-called euchologion, was written on the parchment. Fortunately most of the Archimedean text could be restored.

The method of Archimedes may be called a mechanical infinitesimal method. We must remember that Archimedes was also the founder of statics, and of hydrostatics too. Now his method of integration consists in an application of the principle of the lever to elementary parts of the figure. As an example, we shall see how Archimedes determines the area of a segment of a parabola; this indeed is the first example given in his ἔφοδος.

Let $\widehat{AB}$ be the given segment of the parabola, let R be the midpoint of the chord AB, and draw the diameter d (parallel to the axis) through R, intersecting the arc AB in U (cf. Figure 1). We draw the tangent of the parabola at B and the parallel to d through A. So we obtain $\triangle ABC$ and wish to compare the area

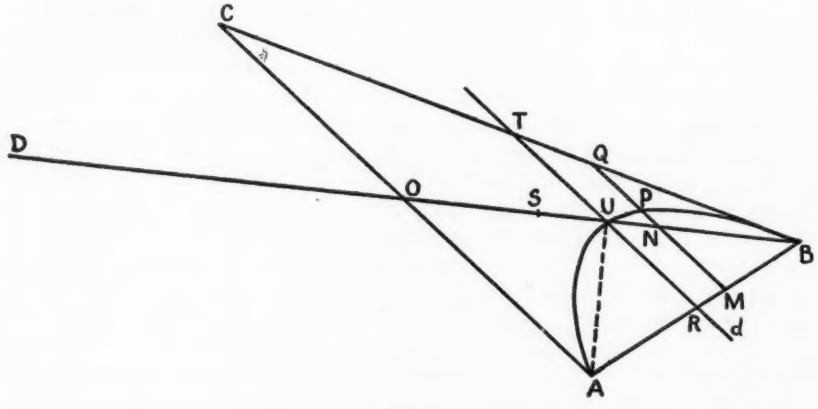

FIG. 1

of the segment $\widehat{AB}$ with the area of this triangle. Let T be the intersection of the tangent BC with the diameter d. From an elementary property of the parabola we obtain: $UT = UR$. Hence the line BU intersects AC in its midpoint O. The centroid S of the $\triangle ABC$ lies on OB with $OS = \frac{1}{3}OB$. Make $OD = OB$. Now through any point P of the arc AUB draw a parallel to the diameter, $MNPQ$. Archimedes proves as a property of the parabola that $MP : MQ = AM : AB$, hence $MP : MQ = ON : OB = ON : OD$. Therefore $MP \times OD = MQ \times ON$. That is, according to the principle of the lever: If DOB is considered to be the bar of a balance with the fulcrum in O (cf. Figure 2), then the line-segment PM suspended at D is in equilibrium with the line-segment MQ which remains, without

any change, suspended at N. Since this holds for every line parallel to d, it follows that the total segment $\widehat{AB}$ of the parabola suspended at D is in equilibrium with the $\triangle ABC$ remaining unchanged or, what amounts to the same thing, the segment $\widehat{AB}$ suspended at D is in equilibrium with the $\triangle ABC$ suspended at its centroid S. Therefore, since $OS=\frac{1}{3}OD$, the segment $\widehat{AB}=\frac{1}{3}\triangle ABC$. Moreover, since $RU=\frac{1}{2}RT=\frac{1}{4}AC$, we have $\triangle ABC=4\triangle AUB$, and hence also segment $\widehat{AUB}=\frac{1}{3}\triangle AUB$.

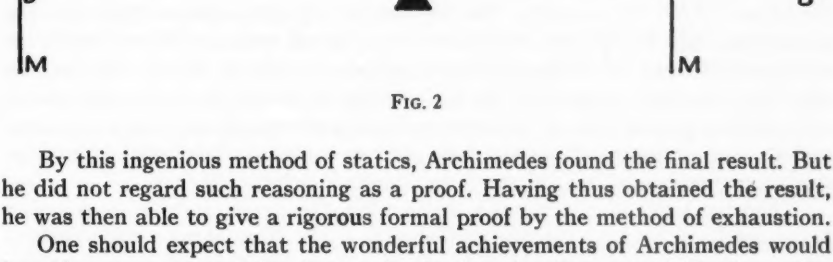

FIG. 2

By this ingenious method of statics, Archimedes found the final result. But he did not regard such reasoning as a proof. Having thus obtained the result, he was then able to give a rigorous formal proof by the method of exhaustion.

One should expect that the wonderful achievements of Archimedes would have become a great stimulus to the further development of Greek mathematics, similar to the great influence of Newton and Leibniz on the mathematical production of succeeding generations. But it is surprising that Archimedes found almost no successors to continue his work. In this connection only one of the subsequent mathematicians is to be mentioned, namely, Dionysodorus who found the volume of the torus. Of course, one has to remember that at the time of Archimedes there lived another outstanding Greek mathematician, Apollonius, about 25 years younger than Archimedes. Apollonius, in a masterly way, completed the Greek theory of conic sections. It is very strange, and I do not understand the reason for it, that soon after Archimedes and Apollonius Greek mathematics declined and that the later development essentially took a different direction. Under the influence of the needs of astronomy, a new branch of mathematics (the roots of which, however, go back also to Archimedes), namely trigonometry, was established and furthermore, much later, the theory of numbers was developed by the work of Diophantus. Original contributions in the direction of Archimedes' work were finally made by one of the latest Greek mathematicians, Pappus (end of the third century A.D.) who stated the important general theorems named after him, in particular, the theorem that the volume of a body of revolution equals the area of the revolving plane figure times the length of the path of the centroid of this area.

When Teutonic tribes, still barbarian at that time, invaded the Roman empire and conquered it, the interest in mathematics almost vanished there; mathematics receded to the Orient, to Byzantium, where at least valuable manu-

scripts were preserved, to Persia, and afterwards to the Arabian countries, where—on the basis of Greek tradition—mathematics flourished in the period about 800–1200 A.D. One of these mathematicians, the Mesopotamian Ibn Al Haitham (about 1000 A.D.), was able to compute the volume of a solid that is generated by rotation of a segment of a parabola about a line perpendicular to its axis.

Under the influence of the Orient, interest in mathematics was slowly awakened in Europe, in particular in the 12th and 13th centuries. As early as the 16th century great discoveries in algebra were made by Italian mathematicians, namely the solution of the algebraic equations of the third and fourth degree. Simultaneously Archimedes' works were studied and understood again.

Then about the beginning of the 17th century the further development of the ideas of Archimedes starts. This was the same great period in which modern science was first established by Galileo. The Flemish engineer Simon Stevin (as early as 1586) and the Italian mathematician Luca Valerio (1604) were the first ones who, by direct passage to the limit, tended to avoid the double *reductio ad absurdum* of the method of exhaustion. Valerio showed directly that the areas under certain curves can be approximated by sums of circumscribed and inscribed rectangles, whose difference can be made arbitrarily small.

Then, in particular, we have to mention the great German astronomer Johannes Kepler who, in 1615, published a book, *Nova stereometria doliorum vinariorum*, on determining the volumes of wine-casks. Somewhat earlier there had been a year of plenty and there was need of barrels for storing the great supply of wine; moreover, Kepler was puzzled by the rules which dealers applied to estimate the approximate contents of a barrel. So he discussed in a popular manner the volumes of various casks and, in particular, asked which cask has the most economic shape. He found that the Austrian barrel was the most economic one. Kepler used the results and methods of Archimedes, but also discussed quite a few new cases. Because of his popular purpose he replaced the rigorous proofs of Archimedes by an intuitive infinitesimal reasoning, in this way stressing the essential points.

Another mathematician of that time had a great influence on further progress; this was the Italian Bonaventura Cavalieri who published in 1635 an important book on the so-called indivisibles, entitled *Geometria indivisibilibus continuorum nova quadam ratione promota*. Indivisibles mean elements of a given dimension which by their motion generate figures of the next higher dimension. Thus a moving point generates a line, a moving line (parallel to a fixed line) generates a plane figure, a moving plane figure (parallel to a fixed plane) generates a solid. Cavalieri speaks, for instance, about "all lines of a plane figure" ("omnes lineae figurae"). Well known is Cavalieri's principle: Two solids (lying between two parallel planes) have the same volume if they intersect each intermediate parallel plane in two equal areas. Cavalieri's views, influenced by late medieval speculations, have somewhat of the spirit of Archimedes' Method, which, however, was not known at that time.

In connection with Cavalieri we must mention also the Swiss Paul Guldin who, besides criticizing Cavalieri, rediscovered Pappus' theorems on bodies of revolution, the Flemish mathematician Gregorius a St. Vincentio who was the first to observe (1647) that the area between a hyperbola and an asymptote behaves like a logarithm, and also the Italian mathematician and physicist Evangelista Torricelli [3] and the French mathematician Gil Persone de Roberval [4]. The important achievements of these last two men will be discussed presently.

About this time another outstanding event occurred in mathematics, the invention of Analytic Geometry by Descartes (1637) and, simultaneously and independently, by Fermat; this invention, of course, had great influence on the development of Calculus. Both Descartes and, in particular, Fermat also made valuable direct contributions to Calculus.

René Descartes, in his *Géométrie*, gave a method of finding the tangents, or rather the normals, to algebraic curves. He draws a circle with center on the x-axis, which cuts the given curve in two points. If these two points coincide, he obtains the normal. Hence the question is reduced to determining double roots of an algebraic equation. Somewhat later, in a letter, Descartes remarked that, instead of circles, intersecting straight lines could also be used for the same purpose.

Fermat's achievements in Calculus were even more important. In fact, he was the greatest mathematician of the first part of the 17th century, not only in general but particularly in the domain of Calculus. Pierre Fermat was a jurist, a councillor of the parliament at Toulouse in southern France. This position left him enough time for intensive mathematical activity. His outstanding work in the theory of numbers is well known. Now, what was *his* method of finding tangents? His procedure was first applied by him to the particular case of determining maxima and minima [5(a)]. He found this method as early as in 1629, communicated it to Descartes in 1638, and had it published in 1642. In order to find the maximum or minimum of an expression, one replaces the unknown A by $A+E$,* and both expressions obtained in this manner are considered approximately equal. One must cancel on both sides all that is possible to cancel. In this way only terms containing E are left. Now divide by E and then drop all terms still containing E. There remains an equation giving that value of A which yields the desired maximum or minimum. That means, if we write $F(A)$ for the given expression, we have to determine A from the equation

$$\left[\frac{F(A+E)-F(A)}{E}\right]_{E=0}=0.$$

This is just our usual method. Of course, the condition is only necessary, but not sufficient for the extreme, and the statement of Fermat yields the result only for polynomials F.

* Fermat always used the letter A for the variable and the letter E for its increment.

Fermat [5(a)] gave at the same time a general method for finding the tangent, in the form of determining the subtangent. Let PT (with T lying on the the x-axis) be the tangent line of the given curve $\mathfrak{C}$ at the point P (cf. Figure 3), let P_1 be a point of $\mathfrak{C}$ in the neighborhood of P, let Q and Q_1 be the projections

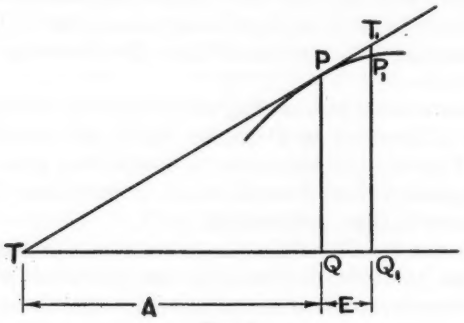

FIG. 3

on the x-axis of P and P_1, respectively, and let T_1 be the point on the tangent line whose projection on the x-axis is Q_1. In order to find the subtangent $A(=TQ)$, whose increment QQ_1 is again designated by E, Fermat uses the similarity of the triangles TQP and TQ_1T_1 and replaces T_1 approximately by P_1. Then he obtains approximately: $A:QP=E:(Q_1P_1-QP)$, that is, in our usual notation if we write the equation of the curve $\mathfrak{C}$ in the form $y=F(x)$,

$$A:F(x) = E:(F(x+E) - F(x));$$

hence

$$A = \frac{F(x)\cdot E}{F(x+E) - F(x)}.$$

Now again one divides the denominator by E and afterwards sets $E=0$.

It should be mentioned here that at this time an entirely different method of constructing tangents of curves, using the parallelogram of velocities, was invented also by Roberval and Torricelli, independently of each other; both of them published it in 1644. On the other hand, somewhat later (before 1659) two Netherlanders Johannes Hudde, for many years mayor of Amsterdam, and René François de Sluse advanced along the road opened up by Descartes and Fermat, giving quite explicit formal rules for finding extremes and subtangents of algebraic curves.

But let us return to Fermat. He had great success also in the theory of integration. He was the first who, by 1636 or earlier, had found and proved the power formula of integration for positive integral exponents n, i.e. a geometrical statement equivalent to the formula which we now write as

$$\int_0^a x^n dx = \frac{a^{n+1}}{n+1}.$$

Roberval also, at the suggestion of Fermat, then found and proved the same theorem. Afterwards Cavalieri discovered it independently,* and he was the first to publish it (1639, 1647); but he proved it explicitly only for the first few cases, including $n=4$, while, as he stated, the general proof which he published was communicated to him by a French mathematician Beaugrand, who quite probably had obtained it from Fermat. At that time Fermat, and then also Torricelli, had already generalized this power formula to rational exponents $n(\neq -1)$. Fermat determined areas under curves which he called "general parabolas" and "general hyperbolas," that is, curves

$$y^m = cx^n, \text{ which, in our notation, leads to } c\int_0^a x^{n/m}dx,$$

and

$$y = \frac{c}{x^m}, \text{ which, in our notation, leads to } c\int_a^{+\infty} \frac{dx}{x^m} \qquad (m > 1).$$

It is remarkable that Fermat in this work does not use subdivisions into equal parts, but subdivisions according to a geometric progression. Other areas were reduced by him to areas under such general parabolas and hyperbolas.

Fermat did interesting work also on the rectification of curves, in a memoir published 1660 [5(b)], where he approximated the arc by segments of tangents, thus using a saw-like figure. At that time various mathematicians obtained rectifications of curves which are now considered classical. In 1645, Torricelli had rectified the logarithmic spiral. The semicubical parabola was rectified independently by the Englishman William Neil (1657), by the Hollander Hendrik van Heuraet, and by Fermat. The rectification of the cycloid was first achieved by the English mathematician and great architect Christopher Wren (1658), and then by Fermat and Roberval, after they had heard of his result. It is noteworthy that Fermat, Neil, van Heuraet, and also Wallis and Huygens, reduced rectifications of curves to the determination of areas of other curves.

Reviewing the achievements of Fermat we see that he was aware of the relation among various problems in differential calculus, and similarly for various problems about definite integrals. But he had not observed the general relation between differentiation and integration.

Another famous French mathematician of that time was Blaise Pascal (a younger friend of Fermat and Roberval), who may be considered a master of integration. Roberval was the first to integrate certain trigonometric functions. Pascal was able to integrate more such trigonometric functions as well as some algebraic functions. As an important means for some of his results Pascal used

* Much later, Blaise Pascal (1654) and Wallis (1656) rediscovered it also.

relations between integrals obtained by interchanging the order of integration in double integrals. Of course, he did this in a geometric form. That is to say, certain volumes were found by means of intersections parallel to one plane and also by intersections parallel to another plane. Since the volume is always the same, Pascal thus obtained a relation between two different integrals, which may be considered to be a kind of integration by parts.

At about the same time important contributions were made by the English mathematician John Wallis, whose book *Arithmetica infinitorum* of 1656 was written in contrast to the geometric work of most of his predecessors. He stressed the notion of the limit. On the other hand, he was very audacious with regard to generalizations and interpolations, but his strong power of intuition kept him on the right track. For instance, he stated the general power formula of integration for any real exponent $n(\neq -1)$.

The notion of limit was carefully considered at that time by the Italian Pietro Mengoli in his book *Geometria speciosa* (1659) [6(a)]. In particular, by modifying the procedure of Luca Valerio (see above), he gave in a precise way a representation of the area under certain special curves as limits of sums of rectangles [6(b)]. Later the same method was also employed by Newton [11, book 1, lemma 2].

One should now mention two other mathematicians who, like Wallis and Mengoli, were contemporary with Newton and Leibniz, but began their work earlier, so that they too are to be considered, at least partially, as predecessors of Newton and Leibniz. One of these men is the great Dutch physicist and mathematician Christiaan Huygens who, among other important results, introduced the notion of evolutes and involutes. It is remarkable that Huygens used to great extent the classical methods of Archimedes, and only for differentiation employed Fermat's method.

The second of these two mathematicians is the Scotsman James Gregory, who like Torricelli and Pascal died in the prime of life, when only 37 years old, and whose genius has found its full recognition only recently [7, 8]. He did excellent work in integration; for instance, in 1668 he published such a difficult result as the following (written in modern notation):

$$\int_0^a \sec x dx = -\log(\sec a - \tan a)$$

and other trigonometric integrals. Moreover, for example, he obtained Newton's interpolation formula, independently of Newton. But perhaps the most important achievements of Gregory belong to the theory of series.

The first great result in the theory of series is due to the German mathematician Nicolaus Mercator (1668), who found the logarithmic series. For this purpose, Mercator used term by term integration of a geometric series. This method was independently discovered, but not published, by Newton. Subsequent contributions were made by William Lord Brouncker, the first president

of the Royal Society in London. Then the most outstanding results concerning infinite series were obtained by Newton and Gregory, who worked essentially independent of each other, though Gregory was influenced by the knowledge of some of Newton's statements, but not of his methods. Both discovered the binomial series and also many series for trigonometric and inverse trigonometric functions. In particular, Gregory found the series for arctan x; a special case of it is the series $\pi/4=1-\frac{1}{3}+\frac{1}{5}-\frac{1}{7}+\cdots$, which was later found independently by Leibniz. Gregory obtained some more complicated series, *e.g.*, for tan x, sec x, log sec x, using processes of differentiation for determining the coefficients of these series, thus anticipating Brook Taylor by more than forty years. This fact has recently been demonstrated by H. W. Turnbull who studied and interpreted notes in Gregory's handwriting (*cf.* [7], pp. 168–176, 350–359). It is obvious that many of the contributions to the theory of series were closely connected with the growth of Calculus.

From all that has been discussed so far we have seen that there certainly was an extensive development of the theory of integration and differentiation in the period immediately before Newton and Leibniz, and that many mathematicans of various nations made great contributions. So we shall ask: What then was missing at that time? One very important point still missing was the general fact that differentiation and integration are inverse processes, that is, the so-called fundamental theorem of integral calculus. It is true that a few mathematicians had already come quite close to this knowledge. First in this connection we must mention Torricelli. In manuscripts left at his death (1647) and published as late as 1919 [cf. 3], he had obtained the distance $s(t)$ of a moving point by means of the quadrature of the velocity $v(t)$, while by the construction of the tangent of the curve $s(t)$ he could recover $v(t)$. It is very doubtful, however, whether he really conceived the significance of this relation. Secondly, as we have already stated, some of the mathematicians of the time knew that the rectification of one curve $\mathfrak{C}_1$ could be obtained by means of the area under another curve $\mathfrak{C}_2$. Fermat in his memoir [5(b)], published in 1660, found the relation between the slopes of these two curves. Moreover, James Gregory, in his book *Geometriae pars universalis* (1668), solved the following problem, which is inverse to the rectification of $\mathfrak{C}_1$, just mentioned: Given a certain curve $\mathfrak{C}_2$, find another curve $\mathfrak{C}_1$ whose length equals the area under $\mathfrak{C}_2$. In the solution of this problem Gregory used the quadrature of an auxiliary curve, which indeed corresponds to obtaining the primitive function of a derivative by means of integration [9]. In spite of this, one again may doubt whether, at that time, he perceived in general that differentiation and integration are inverse operations. The first who made this important discovery in full generality was Isaac Barrow, the teacher of Newton at Cambridge University. Barrow first was professor of Greek language at Cambridge, then he was professor of mathematics first in London and then again at Cambridge. In 1669 he resigned his chair to his pupil Newton, whose superior genius he had recognized. Barrow then devoted the rest of his life to theology. His principal mathematical work was his *Lectiones Geo-*

metricae, published in 1670, together with the second edition of his *Lectiones Opticae*. These geometrical lectures [**10**] contained important systematic contributions to the theory of differentiation and integration, almost all in purely geometric form. Here, for the first time, the inverse character of differentiation and integration was explicitly stated and proved.

At this remarkable point of the development of the theory we must again ask the same question as above: What more remained to be done? The answer is: What had to be created was just the *Calculus*, a general symbolic and systematic method of analytic operations, to be performed by strictly formal rules, independent of the geometric meaning. Now it is just this Calculus which was established by Newton and Leibniz, independent of each other and using different types of symbolism. Newton's first discoveries were made about ten years before those of Leibniz; on the other hand, Leibniz' publications preceded those of Newton, and—what is more important—Leibniz' symbolism, the same as that used by all mathematicians at present, is superior to that of Newton.

Isaac Newton, influenced by his teacher Barrow and also by the work of Wallis, started his "method of fluxions" in the years 1665–1666, his most creative period, at the age of 23 years. Some of his early manuscripts were known to friends of his and indications of his method were contained in some of his letters. He wrote his *Methodus fluxionum et serierum infinitarum* in 1670–1671; but it was not published until 1736, nine years after his death. In his profound work *Philosophiae naturalis principia mathematica* (1687) Newton avoided his method of fluxions, except for a few indications [**11**, book 2, lemma 2], and presented his great discoveries in the classical geometric form, though simplified by using the notion of limits. The first publication of both the method and the notation of Newton is found as late as 1693 in Wallis' works, where Wallis included two of Newton's letters to him. Newton himself then published an account of his method, entitled *Tractatus de quadratura curvarum*,* as an appendix to his *Optics* in 1704.

Newton, considering motions, took the time t as the independent variable, called the dependent variable x "fluent" and its velocity "fluxion," and wrote $\dot{x}$ for the fluxion, i.e. for the derivative with respect to t. The higher derivatives were then designated by $\ddot{x}$, $\dddot{x}$, etc. For the increment of the independent variable t Newton used the letter o and called $\dot{x}o$ (i.e., the differential of x) the "moment" of x. In the case of the inverse process, that is, if the variable x is given as a fluxion, he first designated the fluent (i.e., the antiderivative of x) by $\square\, x$ or $\boxed{x}$, later by $\acute{x}$, and used then for iterated integration $\ddot{x}$, $\dddot{x}$, etc. It has to be stressed that Newton was the first to use systematically the results of differentiation in order to obtain antiderivatives, and hence to evaluate integrals.

Our notations, now used generally in Calculus, are due to Gottfried Wilhelm Leibniz. He was a universal spirit, extremely versatile and interested in every kind of knowledge and scholarship, perhaps most famous as a philosopher. He

* Most of it was already written in 1676.

started as a jurist, was soon active in diplomacy, then became librarian and historian at Hanover, and later (1700) founded the Berlin Academy of Sciences. In mathematics, his first publication, as a graduate student, concerned combinations and permutations; then he soon became interested in the theory of differences and in constructing a computing machine. At the suggestion of Huygens (1673), Leibniz thoroughly studied the works of previous mathematicians on integration and differentiation. In particular, he was much influenced by the work of Pascal. The story of Leibniz' own discoveries can be traced in all details, since his manuscripts with his dated sketches were found at the Hanover library (edited and published by C. I. Gerhardt in the middle of the 19th century) [**12, 13, 14, 14a**]. Leibniz' new notation was first introduced by him on October 29, 1675 [**12,** pp. 121–127]. On this day, as on preceding days, Leibniz discussed integrations using Cavalieri's "omnes lineae." Here he abbreviated "omnes" or "omnia" to "omn." and applied quite a few formal operations to this symbol. Then he remarked: "It will be useful to write $\int$ for omn., thus $\int l$ for omn. l, that is, the sum of those l's." Hence $\int$ is derived from the first letter of the word *summa*. Later in the same manuscript, he came to the "contrary calculus" and continued thus: "If $\int l \sqcap ya$" ($\sqcap$ is the equal sign of Leibniz), "then let us set $l \sqcap ya/d$. Certainly as $\int$ will increase the dimensions, so d will diminish them. $\int$, however, designates a sum, d a difference." Hence Leibniz first wrote the differential sign d into the denominator of the variable. But two or three weeks later he writes dx, dy, dx/dy, and the integrals $\int ydy$ or $\int ydx$. So he arrived at the notation, now classical.

In 1684, Leibniz first published his differential calculus in a paper (issued in the newly founded *Acta Eruditorum*) with the title *Nova methodus pro maximis, itemque tangentibus,* · · · . Here he deals with differentials, and it is noteworthy that he introduces dx as an arbitrary finite interval and then defines dy by the proportion $dy:dx =$ ordinate : subtangent. Then, in 1686, he published also a paper containing his notation of the integral. (The word "integral" was introduced by Jakob Bernoulli, 1690.)

Later Leibniz was accused by friends and followers of Newton of having plagiarized Newton's ideas. This unfortunate and undignified controversy started first in 1699 and became continuously more and more furious, so that Leibniz' last years were filled with bitterness. But considering Leibniz' manuscripts, nobody can doubt at present that Newton and Leibniz founded their Calculus independently.

The invention of Calculus stimulated an immense and energetic further development. On the English side Taylor and Maclaurin, on the continent the eminent Basel mathematicians, first the brothers Bernoulli, then the prodigious Euler, and later the Frenchman D'Alembert and the Italian Lagrange contributed greatly to this development. Moreover, other closely related parts of mathematics soon originated in connection with the Calculus, e.g., the theory of differential equations, the calculus of variations, differential geometry.

While Newton and Leibniz had rather reasonable (although not always con-

sistent) ideas about the fundamentals of the new Calculus, the extremely rapid further development caused the basic concepts to be neglected or to be treated in a very unsatisfactory manner. In particular, Euler is an example of this tendency. A few mathematicians, among them D'Alembert, stressed the necessity of using the notion of the limit as foundation of the Calculus [15]. But it was Cauchy in the beginning of the 19th century who, in such a way, developed the Calculus systematically and consistently. So at last Cauchy and his many successors gave a solid basis to the Calculus.

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COMPUTATION OF THE INVERSE OF A MATRIX

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The computation of the inverse of a matrix lies at the heart of much of matric theory and particularly of its applications. Let us consider a method which simplifies the actual computation of the inverse of a matrix.

The standard formula for computing the inverse of a square matrix A is excellent for theoretical use, but in general practice it involves the computation not only of the determinant of A, but also of all n^2 of its $(n-1)$-rowed minor determinants. This is a laborious task. Burgess* and Albert† have made identical suggestions for simplifying the computation. Their method is discussed in detail in *Elementary Matrices* by Frazer, Duncan & Collar. It is a special case of the method to be presented in this paper.

Let us first prove the following

THEOREM: *If A is a non-singular square matrix, then for any P and Q such that $PAQ=I$, $A^{-1}=QP$.*

If A, P, and Q are non-singular matrices such that

$$PAQ = I,$$

then

$$QPAQ = QI = Q.$$

By multiplying on the right by Q^{-1}, one obtains

$$QPA = I,$$

or

$$QP = A^{-1}.$$

A simple method of obtaining a P and Q, employed in the theorem, will now be developed. It is established easily that if A is carried into I by a series of elementary transformations on rows and columns, and then one obtains a matrix P by applying the same row transformations to I, and a matrix Q by applying the same column transformations to I, these matrices will be a P and Q of the theorem. This is immediate, for P and Q are the unimodular matrices which accomplish the sequence of elementary operations that carry A into I. Thus, by pre- and post-multiplication,

$$\begin{pmatrix} P & PAQ \\ 0 & Q \end{pmatrix} = \begin{pmatrix} P & 0 \\ 0 & I \end{pmatrix}\begin{pmatrix} I & A \\ 0 & I \end{pmatrix}\begin{pmatrix} I & 0 \\ 0 & Q \end{pmatrix} = \begin{pmatrix} P & I \\ 0 & Q \end{pmatrix},$$

and $PAQ=I$ or $A^{-1}=QP$ by the theorem.

* H. T. Burgess, On the Matrix Equation $BX=C$, this MONTHLY, vol. 23, 1916, pp. 152–5.

† A. A. Albert, A Rule for Computing the Inverse of a Matrix, this MONTHLY, vol. 48, 1941, pp. 198–9.

In the actual computation of A^{-1} by this method, the P and Q will contain zeros, and the formation of the product is a simple matter. In fact, the labor involved in computing A^{-1} is usually less than that involved in computing a single element of A^{-1} by the standard theoretical method.

It is even unnecessary to keep track of the transformations made; a much simpler method is to construct an L-shaped array

$$\begin{matrix} I & A \\ & I \end{matrix}$$

and actually perform the transformations on this array. In this way no matric multiplication, and indeed no record of the transformations, need be made to obtain P and Q. To avoid developing theory for the new concept of the L-shaped array we may fill in the lower left corner with zeros, giving a matrix

$$B = \begin{pmatrix} I & A \\ 0 & I \end{pmatrix}.$$

It should be kept in mind that all transformations must occur within A, and that the matrix B is used merely to keep track of the result of these transformations.

In any computational process it is essential that a method exist for checking the process at frequent stages. The above process can be checked at any stage desired. Assume we have obtained

$$\begin{pmatrix} I & A \\ 0 & I \end{pmatrix} \rightarrow \begin{pmatrix} N & A' \\ 0 & M \end{pmatrix}.$$

To check the work one need only verify that $NAM = A'$.

An example may clarify the method. Let us compute A^{-1} where

$$A = \begin{pmatrix} 17 & 44 & 26 \\ -5 & -28 & 9 \\ 8 & 19 & 14 \end{pmatrix}.$$

First form

$$B = \begin{pmatrix} I & A \\ 0 & I \end{pmatrix} = \left(\begin{array}{ccc:ccc} 1 & 0 & 0 & 17 & 44 & 26 \\ 0 & 1 & 0 & -5 & -28 & 9 \\ 0 & 0 & 1 & 8 & 19 & 14 \\ \hdashline & & & 1 & 0 & 0 \\ & 0 & & 0 & 1 & 0 \\ & & & 0 & 0 & 1 \end{array}\right).$$

By row and column transformations on the first n rows and last n columns of

B reduce the upper right hand corner of B (namely A) to I. By working in B we keep track of the result of the transformations made, thus determining a P and Q for our theorem.

Rather than divide the first row (or third column) of B by 17, I have chosen to subtract twice the third row from the first to obtain a 1 in the a_{11} position of A. Thus

$$B \rightarrow \left(\begin{array}{ccc:ccc} 1 & 0 & -2 & 1 & 6 & -2 \\ 0 & 1 & 0 & -5 & -28 & 9 \\ 0 & 0 & 1 & 8 & 19 & 14 \\ \hdashline & & & 1 & 0 & 0 \\ & 0 & & 0 & 1 & 0 \\ & & & 0 & 0 & 1 \end{array}\right)$$

$$\rightarrow \left(\begin{array}{ccc:ccc} 1 & 0 & -2 & 1 & 0 & 0 \\ 5 & 1 & -10 & 0 & 2 & -1 \\ -8 & 0 & 17 & 0 & -29 & 30 \\ \hdashline & & & 1 & -6 & 2 \\ & 0 & & 0 & 1 & 0 \\ & & & 0 & 0 & 1 \end{array}\right).$$

Adding the last column to the next to last, and continuing, one obtains

$$\left(\begin{array}{ccc:ccc} 1 & 0 & -2 & 1 & 0 & 0 \\ 5 & 1 & -10 & 0 & 1 & -1 \\ -8 & 0 & 17 & 0 & 1 & 30 \\ \hdashline & & & 1 & -4 & 2 \\ & 0 & & 0 & 1 & 0 \\ & & & 0 & 1 & 1 \end{array}\right)$$

$$\rightarrow \left(\begin{array}{ccc:ccc} 1 & 0 & -2 & 1 & 0 & 0 \\ 5 & 1 & -10 & 0 & 1 & 0 \\ -13 & -1 & 27 & 0 & 0 & 31 \\ \hdashline & & & 1 & -4 & -2 \\ & 0 & & 0 & 1 & 1 \\ & & & 0 & 1 & 2 \end{array}\right).$$

Upon dividing the third row by 31, one obtains

$$\left(\begin{array}{ccc:ccc} 1 & 0 & -2 & 1 & 0 & 0 \\ 5 & 1 & -10 & 0 & 1 & 0 \\ -\frac{13}{31} & -\frac{1}{31} & \frac{27}{31} & 0 & 0 & 1 \\ \hdashline 0 & 0 & 0 & 1 & -4 & -2 \\ 0 & 0 & 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 & 1 & 2 \end{array}\right) = \begin{pmatrix} P & I \\ 0 & Q \end{pmatrix}.$$

By the use of our theorem,

$$A^{-1} = QP = \begin{pmatrix} 1 & -4 & -2 \\ 0 & 1 & 1 \\ 0 & 1 & 2 \end{pmatrix} \cdot \begin{pmatrix} 1 & 0 & -2 \\ 5 & 1 & -10 \\ -\frac{13}{31} & -\frac{1}{31} & \frac{27}{31} \end{pmatrix}$$

$$= \begin{pmatrix} -\frac{563}{31} & -\frac{122}{31} & \frac{1124}{31} \\ \frac{142}{31} & \frac{30}{31} & -\frac{283}{31} \\ \frac{129}{31} & \frac{29}{31} & -\frac{256}{31} \end{pmatrix}.$$

By using a different sequence of transformations one obtains a different P and Q, but the product QP is again A^{-1}; for example,

$$B \rightarrow \left(\begin{array}{ccc:ccc} 1 & 0 & -2 & 1 & 0 & 0 \\ \frac{5}{2} & \frac{1}{2} & -5 & 0 & 1 & 0 \\ \frac{129}{31} & \frac{29}{31} & -\frac{256}{31} & 0 & 0 & 1 \\ \hdashline 0 & 0 & 0 & 1 & -6 & -1 \\ 0 & 0 & 0 & 0 & 1 & \frac{1}{2} \\ 0 & 0 & 0 & 0 & 0 & 1 \end{array}\right) = \begin{pmatrix} P & I \\ 0 & Q \end{pmatrix},$$

and

$$QP = \begin{pmatrix} 1 & -6 & -1 \\ 0 & 1 & \frac{1}{2} \\ 0 & 0 & 1 \end{pmatrix} \cdot \begin{pmatrix} 1 & 0 & -2 \\ \frac{5}{2} & \frac{1}{2} & -5 \\ \frac{129}{31} & \frac{29}{31} & -\frac{256}{31} \end{pmatrix}$$

$$= \begin{pmatrix} -\frac{563}{31} & -\frac{122}{31} & \frac{1124}{31} \\ \frac{142}{31} & \frac{30}{31} & -\frac{283}{31} \\ \frac{129}{31} & \frac{29}{31} & -\frac{256}{31} \end{pmatrix} = A^{-1}.$$

In the applications of matric theory one must deal frequently with matrices having decimal elements. These are most readily handled by machine computation. The method of this paper is easily adapted to this medium. An algorithm for machine computation will now be obtained. However, the computer who understands the method presented here will experience no difficulty in deviating from this algorithm to take advantage of fortunate combinations.

Step 1: Form B as above. Interchange the first n rows (or last n columns) of B until the first diagonal element of A (a_{11}) is not zero.

Step 2: Multiply the first row by $(a_{11})^{-1}$, giving 1 in the a_{11} position of A.

Step 3: Continuing to do all work in B, add $-a_{12}$ times the first column of A to the second, add $-a_{13}$ times the first column of A to the third, *etc.* Similarly, add $-a_{21}$ times the first row to the second row and $-a_{31}$ times the first row to the third, until we have transformed A into a matrix having 1 in the a_{11} position and zeros in the remainder of the first row and column.

Step 4: Repeat steps 1, 2, and 3 for the minor obtained by disregarding the row and column just worked on.

Step 5: Repeat step 4 until the upper right hand corner of B (namely A) has been transformed into I. By working in B we have automatically obtained the desired P and Q, and $A^{-1} = QP$ may be formed in the usual manner. The process may be checked by the method already suggested.

The method of Burgess mentioned in the second paragraph of this paper is a special case of the method presented here, in which A is transformed into I by row transformations only. In actual practice I have found it quicker to use both row and column transformations, and then form the product QP. An analysis of the work involved in the algorithm of this paper and in the best modifica-

tion (double triangularization) of Burgess' method shows there to be n^3+n multiplications to perform in each case. Thus, in the computation of matrices with random decimal elements, where the work is often done in an entirely unthinking fashion, as if by an automaton, there will be little advantage to the more general method. However, if the computer is alert to advantageous combinations it is clear that there will be more opportunity to take advantage of fortunate combinations using the general method involving both row and column transformations. This is particularly true when some of the elements are integers, or when, in a given column or row, the elements are pre-known factors of one another, or differ by a pre-known amount, as is often the case with actual laboratory data.

SOLUTION OF QUADRATIC EQUATIONS AND TRIANGLES BY MACHINE*

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1. Outline. First, the operations on a computing machine are expressed in symbols. These are applied to the solution of quadratic equations. Square roots are considered as solutions of special quadratic equations. A method is then given for finding ϕ from $\cos\phi$, involving 4 square roots. The accuracy can be increased at pleasure by taking more square roots.

Finally, the methods are applied to the solution of triangles. It is shown how every case of a triangle can be solved without reference to tables. The material is almost wholly original with the author.

There are two main practical consequences of our method of solving triangles. Assuming one has available a 10-place computing machine and a 6-place set of tables, then one can obtain a solution to a triangle which is correct to 9 significant places, instead of only 6. If it were required to build a super-machine for the automatic solution of triangles, use of our method would obviate the necessity of building a set of tables into the machine.

For those wishing to solve a triangle in the least time, paragraph 9 is regarded as providing the answer.

2. Machine operations. The computing machine consists of essentially three parts, the keyboard called K, the accumulator called A, and the counter called C. More elaborate machines include devices by which the interplay between K, A, and C may be made more rapid and automatic. These complications in the machine do not change the essential relation between K, A, and C, so that a consideration of the less elaborate machines will suffice.

* This paper was suggested by the brief note by Prof. H. E. Stelson, this MONTHLY, vol. 56, 1949, pp. 94–95, in which he showed how to solve the case of a triangle, given three sides.

There is essentially only one operation, called $T(1)$. As a result of this operation, the number in K is added to the number in A, and the number in C is increased by 1. The operation $T(1)$ is carried out on the older machines by turning a crank, and on the newer machines by touching the plusbar. If the operation $T(1)$ is carried out n times, the notation $T(n)$ is used. $T(1024)$ does not require the crank to be turned 1024 times, for one turn in the 1000-position of the carriage will carry out $T(1000)$.

The counter C counts how many times $T(1)$ is carried out, and the accumulator A accumulates the totals of the readings of K for the various T's.

There are means by which the readings of A, C, and K can be set at any desired number (up to a certain limited number of digits). The operation of making K read n is $K(n)$. Similarly, $A(n)$ or $C(n)$ means that the number n is to be entered in A or C. It is not necessary to introduce a new notation for clearing A, C, or K, for we need only write $A(0)$, $C(0)$, or $K(0)$.

Division is carried out by repeated subtraction. To find y/x, start with the operations $A(0)$, $K(y)$, $T(1)$, as a consequence of which, the reading of A is y. On some machines the operation $A(y)$ can be carried out directly. Then, $C(0)$, $K(x)$. Now start repeating the operation $T(-1)$ as many times as possible, without making the reading of A become negative. On machines having automatic division, the last step is carried out by throwing the division lever. We shall call this operation "Div." The result is the same whether it is done automatically or by hand.

When performing the operation $T(-n)$, the reading of C would be expected to be negative, and hence hard to read. This can be avoided on most machines by setting the machine for division.

The notation $\Delta K(n)$ means to increase the reading of K by n. In most cases, n will be a small number, like 1 or 3, and the operation is carried out by the mere pressing of a key.

Finally, the notation $Q(a, h)$ stands for an operation used in solving quadratic equations. It consists of three parts or sub-operations:

$$Q(a, h) = \Delta K(ah), \qquad T(h), \qquad \Delta K(ah).$$

3. The solution of quadratic equations. In the equation

$$ax^2 + bx + c = 0, \tag{1}$$

it is supposed that $b \geqq 0$. This may require multiplication by -1. Get the machine ready for the solution of equation (1) by the operations:

$$A(c), \qquad K(b), \qquad C(0),$$

and make a mental note of the value of a. As remarked in connection with division, the operation $A(c)$ may require three steps. If c is negative, $A(c)$ is effected by the three steps, $K(-c)$, $T(-1)$, $K(0)$.

With the machine so prepared, the operation $Q(a, h)$ will have the effect of reducing the roots of the equation by h, as in Horner's Method. Since $Q(a, h)$

can be carried out rapidly, one would as soon carry out $Q(a, 1)$ seven times as to carry out $Q(a, 7)$ once. The effect would be the same.

Example 1. Solve the equation $x^2+3x-11=0$.

Solution. Perform the following operations:

$$K(11), \quad T(-1), \quad C(0), \quad K(3),$$

and make a mental note that $a=1$, the coefficient of x^2. With -11 and 3 well to the left in A and K, mark the decimal points, leaving plenty of spaces at the right for decimals.

Either $Q(1, 2)$, or $Q(1, 1)$, $Q(1, 1)$.

More in detail, $Q(1, 1)$ consists of $\Delta K(1)$, $T(1)$, $\Delta K(1)$. After these three steps, K reads 5, and A reads -7. After the second $Q(1, 1)$, K reads $+7$, A reads -1, and C reads 2. The roots of the equation have now been reduced 2 units.

Next, $Q(1, 0.1)$, $Q(1, 0.04)$, $Q(1, 0.00005)$, and so forth.

Read the root, $x_1=2.14005\cdots$, on C.

$$x_2 = -\ 3 - 2.14005 = -\ 5.14005, \quad \text{or}$$

$$x_2 = -\ 11 \div 2.14005 = -\ 5.140066\cdots.$$

When h is small, the 3-step operation $Q(a, h)$ can be reduced to its middle step, namely $T(h)$, with only a small resulting error. Suppose the operator proceeded as above and found $x_1=2.14005$ on a 10-place machine equipped with automatic division. Then to avoid the negative reading of A, perform the operation $T(0.00001)$, followed by the operation "Div." The resulting reading of C is 2.140054945, correct to 9 decimal places. It is checked by finding x_2 in two ways.

4. Extraction of square roots. To find $\sqrt{N}$, merely solve the quadratic equation $-x^2+0x+N=0$. Note that $a=-1$. Take h negative.

Example 2. Find the square root of 7.

Solution. Prepare the machine for the equation by the operations:

$$A(0), \quad K(7), \quad T(1), \quad K(0), \quad C(0).$$

Next:

$$Q(-1, -2) = \Delta K(2), \quad T(-2), \quad \Delta K(2).$$

If this results in making the reading of C negative, start over, and this time set the machine for division. Then after the operation $Q(-1, -2)$, C will read 2. This is really -2, for we are finding the negative square root of 7.

$$Q(-1, -0.5), \quad Q(-1, -0.1), \quad \text{or} \quad Q(-1, -0.6), \quad \text{or}$$

$$Q(-1, -0.7), \quad Q(-1, +0.1).$$

In other words, one must guess how many tenths to take off. One may take one tenth at a time, or guess 5 tenths at once. If this is not enough, take another. If the operator overshoots and takes off 7 tenths, he may put one back. Whichever he does, he has the same result, and K reads 5.2, C reads 2.6, and A reads 0.24. Then follow the operations:

$Q(-1, -0.04)$, $Q(-1, -0.005)$, and so forth.
The result, $2.645 \cdots$, is read on C.

5. Finding cos $\frac{1}{2}\theta$. One practical application of extracting square roots is in finding cos $\frac{1}{2}\theta$ from cos θ, by the well known formula:

$$\cos \tfrac{1}{2}\theta = \sqrt{\tfrac{1}{2}(1 + \cos \theta)}.$$

This starts with the operations:

$$A(0), \quad K(1 + \cos \theta), \quad T(+0.5), \quad C(0), \quad K(0).$$

The 1 of $1+\cos \theta$ is entered in the extreme left place in K. The 0 of $+0.5$ is in the extreme left place in C. The machine is now ready for the extraction of the square root whose value is cos $\frac{1}{2}\theta$.

In counting the significant figures in cos θ, initial 9's should not be counted. If $\cos \theta = 0.9999365$, then one has only 3 significant places. It is desirable not to lose significant places in the process of finding cos $\frac{1}{2}\theta$.

Example 3. If $\cos \theta = 0.98713251$, find cos $\frac{1}{2}\theta$.

Solution. We shall suppose that there is available an 8-place machine, but as one place is used for the units place, it gives cosines to 7 significant places. In the present example, the initial 9 makes difficulties, as one must go out one more place to have 7 significant places.

$$C(0), \quad K(1.9871325), \quad T(+0.5), \quad K(0.0000001), \quad T(+0.05), \quad K(0), \quad A(0)$$

The machine is now ready to extract the square root of $\frac{1}{2}(1+\cos \theta)$. The machine gives the square root as 0.9967779, but stops there since it is only an 8 place machine. K reads $1.9935 \cdots$, or practically 2. A reads $\cdots 7307 \cdots$. The next two places of the desired square root can be read by inspection. It requires merely dividing $\cdots 7307 \cdots$ by 2, namely, 36 or 37. Therefore, on an 8-place machine, one finds readily that $\cos \frac{1}{2}\theta = 0.996777937$, to the full 7 significant places.

Of course, it would have been easier to use a 10-place machine, but with a 10-place machine, one would expect 9 significant places. The procedure of Example 3 illustrates how one finds cos $\frac{1}{2}\theta$ to the full $n-1$ significant places on an n-place machine.

6. Finding ϕ from cos ϕ. From Taylor's series, we have

$$\cos \phi = 1 - \frac{\phi^2}{2} + \frac{\phi^4}{24} - \frac{\phi^6}{720} + \cdots \tag{2}$$

$$\cos \tfrac{1}{2}\phi = 1 - \frac{\phi^2}{4 \cdot 2} + \frac{\phi^4}{4^2 \cdot 24} - \frac{\phi^6}{4^3 \cdot 720} + \cdots \tag{3}$$

$$\cos \tfrac{1}{4}\phi = 1 - \frac{\phi^2}{4^2 \cdot 2} + \frac{\phi^4}{4^4 \cdot 24} - \frac{\phi^6}{4^6 \cdot 720} + \cdots . \tag{4}$$

The term in ϕ^4 is readily eliminated between (2) and (3). The term in ϕ^6 may either be dropped, or carried as an approximation to the error. The resulting equation is solved for ϕ^2, and ϕ is found by extracting one square root. The resulting formula is (5) given below. Similarly, working with (2), (3) and (4), the terms in ϕ^4 and ϕ^6 may both be eliminated, and Formula (6) is derived.

$$(5) \qquad \phi = \sqrt{\frac{15 + \cos\phi - 16\cos\tfrac{1}{2}\phi}{1.5}} + \frac{\phi^5}{2880} + \cdots .$$

$$(6) \qquad \phi = \sqrt{\frac{945 - \cos\phi + 80\cos\tfrac{1}{2}\phi - 1024\cos\tfrac{1}{4}\phi}{22.5}} + \frac{\phi^7}{2^7 \cdot 8!} + \cdots .$$

When computing with (5) or (6), the final terms are computed only roughly, to see how many significant places are given by the radical. The remainder term in (5) happens to be precisely the same as that of the formula given by Prof. H. E. Stelson.

Example 4. Find the largest angle in a triangle with sides 7, 9, and 14. (Problem taken from H. E. Stelson's paper.)

Solution. By the law of cosines, $\cos\theta = -66/126$.

t	$\cos\theta/t$
1	−0.523809524
2	+0.487950037
4	0.862539865
8	0.9650232808
16	0.99121725185
32	0.99780189713

Since a 10-place machine was used, all cosines are carried to 9 significant places. In (6), take $\phi = \theta/8$. Then

$$\theta = 8(0.2652641622 \cdots) = 2.122113298.$$

With $\phi = 0.26 \cdots$, $\phi^7/2^7.8! = \phi^7/5{,}000{,}000$, which vanishes to 10 decimal places. Had ϕ been taken as $\theta = 2.12 \cdots$, the error would have been much larger.

7. Conversion of angles to degrees. For machine computation, it is more convenient to have all angles in radians rather than in degrees, minutes, and seconds. Machines are not designed to handle addition of numbers in the sexagesimal system. For the computation of $\sin x$, $\cos x$, and $\tan x$ from the series, it is best to have x in radians. But suppose it is desired to express an angle in the Babylonian manner. The following procedure is suggested:

Let D, M, and S be, respectively, the number of radians in a degree, in a

minute, and in a second. Then

$$D = 0.01745329252\cdots$$
$$M = 0.0002908882087\cdots$$
$$S = 0.000004848136811\cdots.$$

Example 5. Convert 2.122113298 radians to degrees, minutes, and seconds.

Solution. Since θ is given to 10 places, we suppose a 10-place computing machine. Perform the following operations:

$$A(0), \quad K(\theta = 2.122113298), \quad T(1.000000000), \quad C(0).$$
$$K(D = 1745329252), \quad T(-121.0000000).$$

In other words, carry out the division, so long as the quotient, here 121, remains an integer. Do not clear A, K, or C until directed to.

$$K(M = 2908882087), \quad T(-00035.00000).$$
$$K(S = 4848136811), \quad T(-0000017.288).$$
$$\theta = 121^\circ\ 35'\ 17.288''\ \text{as read from } C.$$

In a similar manner, the angle $121^\circ\ 35'\ 17.288''$ can be reconverted to radians, by the use of D, M, and S.

8. Solution of triangles. The procedure for the solution without tables of any triangle is now obvious.

Given the three sides, apply the Law of Cosines, and find each angle from its cosine. Check by the relation $A+B+C=\pi$.

Given two sides and the included angle, a, b, and C, find cos C by the cosine series, and c from the Law of Cosines.

Given the ambiguous case, a, b, and A, one procedure is to start with the Law of Sines. Find sin A by the sine series, sin B by the Law of Sines. Then sin B $=\cos(\frac{1}{2}\pi - B)$, and $\frac{1}{2}\pi - B$ can be found from its cosine, as in Example 4. As the sign of $(\frac{1}{2}\pi - B)$ is in doubt, two values of B result.

Or, if only the value of c is desired, use the Law of Cosines immediately,

$$a^2 = c^2 + b^2 - 2bc\cos A.$$

This is a quadratic equation in c. It is solved as in Example 1.

Example 6. Given $a=12.02$, $b=12.83$, and $A=1.1023$ radians, find c.

Solution. From the Law of Cosines,

$$12.02^2 = c^2 + 12.83^2 - (2)(12.83)(\cos 1.1023)(c).$$

Cos $1.1023=0.451545147$, as can be found directly from the cosine series, requiring only 7 terms to give 9 places, or from the more rapidly convergent series sin $(\frac{1}{2}\pi - 1.1023)$.

$$-c^2 + 11.5866485c - 20.1285 = 0.$$

$$c_1 = 2.128067661.$$

$$c_2 = 11.5866485 - c_1 = 9.458580839,$$

or

$$c_2 = 20.1285 \div c_1 = 9.458580838.$$

The case of two angles and one side is handled by the Law of Sines, with the sines of the three angles found by series.

9. Solution of triangles with machine and tables. For the quickest possible method of solving triangles, use both tables and machine. Follow the methods of Section 8, and whenever it is necessary to find a trigonometric function of a given angle, or to find the value of the angle from one of the trigonometric functions, use the tables. The machine will be found helpful in the interpolation.

With the necessary multiplying, squaring, extracting the square root, and other computation done on the machine, every triangle can be solved with 3 applications to the tables. In some cases, this includes the check, in other cases, a fourth reference to the tables will be required for a check. This contrasts with 8 applications to the tables, when the computation is done by logarithms instead of machine.

MATHEMATICAL NOTES

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ON A CONJECTURE OF KLEE

PAUL ERDÖS, University of Michigan

1. Introduction. Klee[1] denotes by $S_k(m)$ the number of solutions of $\phi(x) = m$, where x has exactly k prime factors which appear to the first power in the factorization of x. Lampek[1] observed that

$$\phi\left(\frac{(n!)^2}{\phi(n!)}\right) = n!.$$

Klee[1] remarks that except for the prime 2, all prime factors of $n!/\phi(n!)$ are multiple. Thus $S_0(n!) > 0$. Klee[1] conjectures that for all n, $S_1(n!) > 0$. Gupta[2] recently proved this conjecture, in fact he proved that $\lim_{n=\infty} S_1(n!) = \infty$. In the present note we prove that $\lim_{n=\infty} S_k(n!) = \infty$ for every k, and state without

[1] This MONTHLY, vol. 56 (1949), pp. 21–26.

[2] *Ibid.*, vol. 57 (1950), pp. 326–329.

proof a few other problems and results.

2. Lemmas. First we prove three lemmas.

Lemma 1. *Let* $b \mid a$, *and assume that* a/b *has the same prime factors as* a (*that is, all the prime factors of* b *occur in* a *with a higher exponent*). *Then*

$$\phi\left(\frac{a}{b}\right) = \frac{\phi(a)}{b}.$$

This follows immediately from the definition of the ϕ-function.

Lemma 2. *The number of primes* q, $n<q<2n$, $q\equiv 1 \pmod 6$, *is greater than* $c_1 n/\log n$ *for a suitable constant* c_1 *and sufficiently large* n.

This follows immediately from the prime number theorem for arithmetic progressions (or also from a more elementary result).[3]

Lemma 3. *Let* n *be sufficiently large. Put*

$$A_{q_1,q_2,\cdots,q_k} = \frac{n!}{\prod (p-1)\cdot(q_1-1)(q_2-1)\cdots(q_k-1)},$$

where p *runs through the primes* $\leqq n$ *and* $n<q<2n$, $q\equiv 1 \pmod 6$. *Then* $A_{q_1,\cdots,q_k}$ *is an integer, and* $p \mid A_{q_1,\cdots,q_k}$, *for* $p \leqq n$.

First of all from Lemma 2, for sufficiently large n the number of q's is $>c_1 n/\log n > k$; thus $A_{q_1,\cdots,q_k}$ is defined. Let t be a prime. For $n/2<t\leqq n$, $t \mid A_{q_1,\cdots,q_k}$ since $t \mid n!$ while $t \neq p-1$, $q_i-1 \not\equiv 0 \pmod t$ (since $q_i-1\equiv 0 \pmod 6$). Let next $3<t\leqq n/2$. The denominator of $A_{q_1,\cdots,q_k}$ can be written as

$$2^v \prod \frac{p-1}{2} \prod_{i=1}^{k} \frac{q_i-1}{2},$$

and here all the factors are distinct integers $\leqq n$. But t and $2t$ are never of the form $(q_i-1)/2$ (since $(q_i-1)/2\equiv 0 \pmod 3$). Further not both t and $2t$ can be of the form $(p-1)/2$, since either $2t+1$ or $4t+1$ is a multiple of 3. Thus any $3<t\leqq n/2$ occurs with a higher exponent in $n!$ than in the denominator of $A_{q_1,\cdots,q_k}$. If $t=3$ and $n\geqq 12$, then $3 \mid A_{q_1,\cdots,q_k}$, since $12 \neq (p-1)/2$, $12 \neq (q_i-1)/2$. Let now $t=2$. The even numbers $6u+2$ are clearly not of the form $p-1$ ($6u+3 \equiv 0 \pmod 3$). Thus $n!/\prod(p-1)$ is a multiple of $2^{[(n-2)/6]}$. If $\prod_{i=1}^{k}(q_i-1)$ is a multiple of 2^u we clearly have $2^u < 2^k n^k$ and, for sufficiently large n,

$$2^{[(n-2)/6]} > (2n)^k.$$

Thus $2 \mid A_{q_1,\cdots,q_k}$, and Lemma 3 is proved.

3. Theorem. We shall establish the following result.

[3] R. Breusch, Math. Zeitschrift, vol. 34 (1932), pp. 505–526; see also P. Erdos, *ibid.*, vol. 39 (1935), ppl 473–491.

THEOREM. *For sufficiently large n, we have*

$$S_k(n!) > c_2 \frac{n^k}{(\log n)^k}.$$

Proof: We have, by Lemmas 1 and 3, with

$$B_{q_1,\cdots,q_k} = \prod_{i=1}^{k} q_i \frac{(n!)^2}{\phi(n!)\prod_{i=1}^{k}(q_i-1)},$$

$$\phi(B_{q_1,\cdots,q_k}) = \phi\left(\prod_{p\leqq n} p \prod_{i=1}^{k} q_i \frac{n!}{\prod_{p\leqq n}(p-1)\prod_{i=1}^{k}(q_i-1)}\right)$$

$$= \phi\left(\prod_{p\leqq n} p \prod_{i=1}^{k} q_i A_{q_1,\cdots,q_k}\right) = n!.$$

It follows from Lemma 2 that there are more than $c_2 n^k/(\log n)^k$ choices for $q_1, \cdots, q_k$; also, by Lemma 3, $B_{q_1,\cdots,q_k}$ contains exactly k prime factors which appear to the first power in the factorization of $B_{q_1,\cdots,q_k}$. This completes the proof of the Theorem.

4. Further questions. One can ask the question how large has n to be in order that $S_k(n!)>0$. Our proof gives that n has to be greater than $c_3 k \log k$. By a more complicated argument we can show that for a suitable constant c_4, we have $\phi_k|[c_4 k]!|>0$. It is probable that for every $\epsilon>0$ and sufficiently large n we have $\phi_k([(1+\epsilon)k]!)>0$. It is easy to see that $S_n(n!)=0$ for $n>2$.

We can also show that $\lim_{n=\infty} S_k(n!)^{1/n}=1$. On the other hand there exists an absolute constant c_5 so that the number of solutions of $\phi(x)=n!$ is greater than $(n!)^{c_5}$. Previously it was known that there are infinitely many integers m, so that the number of solutions of $\phi(x)=m$ is greater than m^{c_5}. It is an open question whether c_5 can be chosen arbitrarily close to 1.

It seems a difficult question to decide whether $\phi(x)=n!$ is always solvable in squarefree integers x. Similarly it seems difficult to decide whether for sufficiently large n, the equation $\sigma(x)=n!$ is solvable ($\sigma(x)$ denotes the sum of the divisors of x).

If one wants to prove Gupta's[2] result, $S_1(n!)>0$ for all n, it suffices to remark that for $n\geqq 4$ there always is a prime $q\equiv 1 \pmod 6$ in the interval $(n, 2n)$.[3] Also that for $n\geqq 8$,

$$2^{[(n-2)/6]+2} \Big| \frac{n!}{\prod(p-1)}$$

(since 8 contains 2 with exponent 3). Further since $q\leqq 2n$, $q\equiv 1 \pmod 6$, if $2^u|(q-1)$ we have $2^u<2n/3$. Thus if

$$2^{[(n-2)/6]} > n/6,$$

then $S_1(n!) > 0$, and this holds for $n \geqq 14$. For $n < 14$, the relation $S_1(n!) > 0$ can be shown by a short computation. By a slightly longer computation we can show that $S_2(n!) > 0$ for all $n \geqq 2$.

PERFECT SQUARES OF SPECIAL FORM*

VICTOR THÉBAULT, Tennie, Sarthe, France

1. Introduction. This note carries further** the determination of systems of numeration in which there exist pairs of perfect squares having the form

$$aabb = (cc)^2, \qquad bbaa = (dd)^2.$$

2. Necessary and sufficient conditions. It is easy to show† that necessary and sufficient conditions for the above are:

(1) $a + b = B + 1,$ (2) $1 \leqq a, b, c < B,$

(3) $c^2 = a(B - 1) + 1,$ (4) $d^2 = b(B - 1) + 1,$

where B, a, b, c are positive integers.

3. Special cases. The form of (3) suggests an examination of the special cases where

(5) $$c = ma \pm 1,$$

with m an arbitrary positive integer.

From (1), (3) and (5), the following equations result immediately:

(6) $$B = m(ma \pm 2) + 1,$$

(7) $$b = (m \pm 1)[(m \mp 1)a \pm 2].$$

Now (4), (6) and (7) combine to give

(8) $$d^2 = m(m \pm 1)(ma \pm 2)[(m \pm 1)a \pm 2] + 1.$$

This equation is satisfied by $a = 0$, $d = 2m \pm 1$, and by $a = 4$, $d = 4m^2 \pm 2m - 1$. Hence, since the coefficient of a^2 is not a square, there will be infinitely many solutions for each positive integer m (and for many fractional values of m as well).

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* Translated (and abridged) from the French by E. P. Starke, Rutgers University.

** See V. Thébault, Mathesis, 1936 (Supplement); This MONTHLY, *Two Classes of Remarkable Perfect Square Pairs*, 1949, pp. 443–448.

† Or see V. Thébault, Mathesis, *loc. cit.*

4. Examples. The following examples sufficiently illustrate the procedure.

(i) Take $m=1$, $c=a+1$. Then $B=a+3$, $b=B+1-a=4$, and $d^2=4B-3$. Since this last is an odd integer, d^2 may be set equal to $(2k+1)^2$, whence $B=k^2+k+1$. This gives the squares

$$\overline{B-3}\ \overline{B-3}\ 4\ 4=(\overline{B-2}\ \overline{B-2})^2,$$

$$4\ 4\ \overline{B-3}\ \overline{B-3}=(\overline{2k+1}\ \overline{2k+1})^2,$$

to which we have previously called attention.*

(ii) With the values $m=2$ and $c=2a+1$, equation (8) may be written as

$$d^2-3(2a+3)^2=-2,$$

of which the solutions are given by the recurrence relations

$$d_{n+1}=4d_n-d_{n-1}, \qquad a_{n+1}=4a_n-a_{n-1}+3,$$

with $d_0=5$, $a_0=0$; $d_1=19$, $a_1=4$. Thus we have an infinite sequence of pairs of squares of the desired form given by

$$a=a_n, \qquad b=3a_n+6, \qquad c=2a_n+1, \qquad B=4a_n+5, \qquad d=d_n.$$

(iii) With $m=3$, $c=3a-1$, equation (8) becomes

$$(12a-7)^2-2d^2=-1.$$

The solutions of $x^2-2d^2=-1$ are given by

$$x_{n+1}=6x_n-x_{n-1}, \qquad d_{n+1}=6d_n-d_{n-1},$$

with $x_0=1$, $d_0=1$; $x_1=7$, $d_1=5$. It is easy to show that a_n, being equal to $(x_n+7)/12$, is an integer for every n for which $n-2$ is a multiple of 4. The desired squares are now given by $n=4k+2$ and

$$a=(x_n+7)/12, \quad b=(2x_n+2)/3, \quad c=(x_n+3)/4, \quad B=(3x_n+1)/4, \quad d=d_n.$$

(iv) For $m=3/2$, $c=(3a-2)/2$, equation (8) may be reduced to

$$16d^2=45a^2-96a+64,$$

which in turn is reduced, by substitution of $a=(4x+16)/15$, to

$$x^2-5d^2=-4.$$

The solutions of this last equation may be obtained in the form

$$x_{n+1}=18x_n-x_{n-1}, \qquad y_{n+1}=18y_n-y_{n-1},$$

and thus squares of the desired form may be found from

$$a=(4x_n+16)/15, \qquad b=(x_n+1)/3, \qquad c=(2x_n+3)/5,$$
$$B=(3x_n+2)/5, \qquad d=d_n;$$

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* Mathesis, 1936, *loc. cit.*

these will be all integers for x_n, d_n in either of the sequences

$$x_0 = 1, \quad d_0 = 1; \quad x_1 = 29, \quad d_1 = 13; \cdots; \quad \text{with} \quad n = 4k + 2,$$

or

$$x_0 = 4, \quad d_0 = 2; \quad x_1 = 76, \quad d_1 = 34; \cdots; \quad \text{with} \quad n = 4k + 3.$$

(v) Following a suggestion by M. A. Buquet, we easily verify that the polynomials

$$a = 4k, \qquad k = 8m^3 + 4m^2 - 4m - 1, \qquad d = 2k(2m^2 - 1) - 2m - 1$$

reduce (8) to an identity in m. Therefore a and d, together with

$$b = 4(m^2 - 1)k + 2m + 2, \qquad c = 4mk + 1, \qquad B = 4m^2k + 2m + 1,$$

form squares of the desired form for every integral value of m.

5. Table of values. A table is appended showing a few of the simplest examples obtained by these methods. (We omit those in (i) and those which could be obtained from them upon interchange of a and b, and c and d.)

B	a	b	c	d
81	19	63	39	71
177	45	133	89	153
305	75	231	151	265
313	140	174	209	233
1141	284	858	569	989
2461	616	1846	1231	2131
4261	1064	3198	2129	3691
8611	956	7656	2869	8119

ON A GENERALIZATION OF FEUERBACH'S THEOREM

R. GOORMAGHTIGH, Bruges, Belgium

Using complex coordinates, H. F. Sandham has recently (this MONTHLY, [1949, 620–622]) given a remarkable generalization of Feuerbach's theorem by showing that *the angle between the nine-point circle of a triangle ABC and the pedal circle of two isogonal conjugate points P and Q is equal to the angle which the segment PQ subtends at the inverse of either P or Q in the circumcircle of ABC.* A very simple proof of this generalization can be derived from properties of the orthopole and Aiyar's theorem.

Let O be the circumcenter of triangle ABC. Since the orthopole of any circumdiameter lies on the pedal circle of any point of that diameter,[1] the intersections, M and M', of the nine-point circle, having center F, and the pedal circle of P and Q, having for center the midpoint ω of PQ, are the orthopoles, with respect to triangle ABC, of OP and OQ. It is known[2] that $\sphericalangle MFM' = 2\sphericalangle POQ$, and therefore $\sphericalangle MF\omega = \sphericalangle POQ$. But, by Aiyar's theorem,[3] if R is

[1] Gallatly, Modern geometry of the triangle, second edition, p. 52.

[2] *Ibid.*, p. 31.

[3] *Ibid.*, p. 79.

the circumradius, $(OP)(OQ)=2R(\omega F)$. Hence, if Q' is the inverse of Q in the circumcircle of ABC, $\omega F/FM=PO/OQ'$. The triangles ωFM and POQ' are thus similar, and the theorem is established.

A CORRECTION

B. H. ARNOLD, Oregon State College, and IVAN NIVEN, University of Oregon

We regret to report that our notes in this MONTHLY [vol. 56, pp. 465–466 (1949) and vol. 57, pp. 246–248 (1950)] contain a basic error. The mappings employed are not continuous. In the first paper the function $g(z)$ for $|z|\leqq 1$ has a discontinuity along the positive real axis. Similarly, in the second paper the function $g(z)$ in equation (8) is discontinuous. No easy remedy seems to be available to avoid this difficulty, which was pointed out to us by J. C. Oxtoby.

CLASSROOM NOTES

EDITED BY C. B. ALLENDOERFER, Haverford College

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INTEGRATION BY LONG DIVISION

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The familiar relation that $\int f(x)\cdot dx=(1/d)f(x)$, used in symbolic operators, can be introduced in elementary calculus for the purpose of integration.

In the process of long division, the first term in the quotient is an approximation to the quotient. The approximation is *usually* found by considering only the first terms of the dividend and divisor. *When dividing by d*, meaning the differential, the approximation is formed by dividing by d assuming that all the first term except its first factor is constant.

The procedure will give every result that can be found by the method of integration by parts. In fact the formula for integration by parts may be obtained by the following division:

$$\begin{array}{lll} d) & u\cdot dv & (uv \\ & \underline{u\cdot dv+v\cdot du} & \\ & -v\cdot du & \end{array}$$

The quotient, uv, is obtained by dividing $u\cdot dv$ by d, assuming u constant. The remainder, $-v\cdot du$, still remains to be divided by d, with the result: $-\int v\cdot du$.

For example, to divide $x^2\cdot\sin x\cdot dx$ by d, assume that x^2 is constant for the

purpose of obtaining a simple approximation, namely $x^2(1/d)\sin x\cdot dx = -x^2\cos x$. This is taken as the first term in the quotient. The long division then appears as follows:

$$\begin{array}{lll} d) & x^2\cdot\sin x\cdot dx & (-x^2\cdot\cos x + 2x\cdot\sin x + 2\cdot\cos x \\ & \underline{x^2\cdot\sin x\cdot dx - 2x\cdot\cos x\cdot dx} & \\ & \qquad + 2x\cdot\cos x\cdot dx & \\ & \qquad \underline{2x\cdot\cos x\cdot dx + 2\cdot\sin x\cdot dx} & \\ & \qquad\qquad - 2\cdot\sin x\cdot dx & \\ & \qquad\qquad \underline{- 2\cdot\sin x\cdot dx} & \\ & \qquad\qquad 0 & \end{array}$$

When the remainder comes out 0, we know that the division, or rather the integration, is exact. The quotient is the desired integral of $x^2\cdot\sin x$.

In the division of $e^{2x}\cdot\sin 3x\cdot dx$ by d, the remainder never comes out 0, but after two steps the remainder is a constant times the dividend. The fundamental relation between D=dividend, d=divisor and also differential, Q=quotient, and R=remainder, is

$$D = dQ + R.$$

If

$$D = dQ + kD,$$

then

$$(1 - k)D = dQ, \quad \text{and}$$

$$D = d\left(\frac{Q}{1 - k}\right).$$

The result is $Q/(1-k)$, and the integration is seen to be exact.

The long division for $e^{2x}\cdot\sin 3x\cdot dx$ by d appears as follows:

$$\begin{array}{lll} d) & e^{2x}\cdot\sin 3x\cdot dx & (-\tfrac{1}{3}e^{2x}\cdot\cos 3x + \tfrac{2}{9}e^{2x}\cdot\sin 3x \\ & \underline{e^{2x}\cdot\sin 3x\cdot dx - \tfrac{2}{3}e^{2x}\cdot\cos 3x\cdot dx} & \\ & \qquad \tfrac{2}{3}e^{2x}\cdot\cos 3x\cdot dx & \\ & \qquad \underline{\tfrac{2}{3}e^{2x}\cdot\cos 3x\cdot dx + \tfrac{4}{9}e^{2x}\cdot\sin 3x\cdot dx} & \\ & \qquad\qquad - \tfrac{4}{9}e^{2x}\cdot\sin 3x\cdot dx & \end{array}$$

Here $R = -\frac{4}{9}D$, so that $k = -\frac{4}{9}$.

The desired integral is $Q/(1-k)$, namely:

$$\frac{-\tfrac{1}{3}e^{2x}\cdot\cos 3x + \tfrac{2}{9}e^{2x}\cdot\sin 3x}{1 + \tfrac{4}{9}}.$$

PARAMETRIC EQUATIONS AND MECHANICAL MANIPULATION OF MATHEMATICAL SYMBOLS

A. D. FLESHLER, Champlain College

Many writers have pointed out the necessity of impressing upon the student the fact that mathematical symbols and formulae are not to be handled mechanically, expecting that an apparent automatic, correct operation will always yield a correct result. Examples illustrating the pitfalls involved in such a mechanical procedure are legion.* Even the better student who knows his integration and trigonometric transformations will perform the following operation:

$$\int_0^{4\pi} \sqrt{1-\cos x}\,dx = \sqrt{2}\int_0^{4\pi} \sin\frac{x}{2}\,dx = 0(!!).$$

Most students, when the difficulty involved in the above operation is pointed out to them, still remain puzzled why an apparent correct transformation led to the wrong result.

I found that one of the most effective ways of training the student to be on his guard against wrong results arising from such automatic transformations is to ask him to plot a curve given in parametric equations: first by eliminating the parameter and plotting $y=f(x)$, and then plotting directly from the parametric equations $x=h(t)$; $y=g(t)$. The equations, of course, are chosen so that the two methods of plotting lead to two different loci. I gave the class the simple set of equations:

$$x = \sin t; \qquad y = \sin t$$

and I asked them for the locus represented by these equations. The unanimous agreement was that since $x=y$ by eliminating $\sin t$, the locus was a 45 degree line through the origin. I then asked them to plot directly by making a table of values of t, x, y. They quickly realized that they could not get out of the intervals: $-1\leqq x\leqq 1$; $-1\leqq y\leqq 1$ and that the parametric equations represented only a finite portion of the line. I gave them innumerable variations of the above set of equations. In each case a mechanical elimination led to $x=y$; while the original set of equations represented different portions of the line.

For example: The equations $x=e^t$; $y=e^t$ represent the portion of the line in the first quadrant; while the equations: $x=\coth(t)$; $y=\coth(t)$ represent the entire line except the interval: $-1<x<1$; $-1<y<1$. Once the students grasped the significance of the above illustrations they were on their guard against such fallacies when the above method was applied to other loci.

Time and again the student had used the following elimination: Given $x=r\cos\theta$; $y=r\sin\theta$. Squaring and adding he obtained $x^2+y^2=r^2$ a circle. When I gave them the following set of equations $x=a(\tanh t)$; $y=a(\operatorname{sech} t)$ a few were

* See F. S. Nowlan: "Objectives in Teaching College Mathematics," this MONTHLY, vol. 57, pp. 73–82, 1950.

tempted to say that the equations represented a full circle, since, following the above procedure we obtain:

$$x^2 + y^2 = a^2(\text{sech } t)^2 + a^2(\tanh t)^2 = a^2$$

The majority however, realized that y must always be positive; and hence the entire locus must be above the x axis.

The advantage of this method is that once the student grasps the idea he gets real pleasure out of constructing his own illustration and is duly impressed with the importance of a proper interpretation of mathematical symbols and expressions.

AN IMPROVEMENT ON DICKSON'S "BEST METHOD FOR INTEGRAL ROOTS"

J. G. CAMPBELL, Albany, Kentucky

In chapter III, section 20 of L. E. Dickson: "New First Course In the Theory Of Equations," one finds Dickson's "Best Method For Integral Roots." We present the following simplification of the procedure for applying his method. Let $f(x) \equiv a_0x^N + \cdots a_N$ be a polynomial with integral coefficients. We seek its integral roots, r.

THEOREM. *If r is an integral root of $f(x) = 0$ and if m is any other integer not a root, then $r - m$ is a divisor of $f(m)$.*

Proof. Consider the factor theorem. If r is a root of $f(x) = 0$ then $f(x) = (x - r)Q(x)$. Put $x = m$, $f(m) = (m - r)Q(m)$. Hence $r - m$ is a divisor of $f(m)$.

Let $(D_1, D_2, \cdots, D_j)$ be the total set of divisors of $f(m)$. Form the subset $(D_1 + m, D_2 + m, \cdots, D_j + m)$. The root r is contained in this subset, since $(r - m) + m = r$ and $r - m$ is one of the divisors of the first set of numbers.

Now consider the divisors of $f(m_1)$ where m_1 is another integer not a root. Let the total set of divisors be $(d_1, d_2, \cdots, d_k)$. Form the subset $(d_1 + m_1, d_2 + m_1, \cdots, d_k + m_1)$. The root r is contained in this subset of numbers. Let $(s_1, s_2, \cdots, s_l)$ be the set of numbers common to both subsets. r is contained in this set.

To illustrate his method, Dickson gives the following problem in the above mentioned section of his book.

Example. Find the integral roots of $f(x) = x^3 - 20x^2 + 164x - 400 = 0$. The constant term has 30 divisors. There is no negative root and 21 is an upper limit to the roots. Let $m = 0$, then $f(0) = -400$. The divisors less than 21 are (1, 2, 4, 8, 16, 5, 10, 20). The subset is the same since any number plus zero is that number. Now let $m_1 = 1$ and note that $f(m_1) = -255$. The divisors are 3, 5, 17. The subset is $(3+1, 5+1, 17+1)$ or (4, 6, 18). The set common to both subsets is (4). We see that $f(4) = 0$ and 4 is the only integral root. In actual practice it is possibly better to consider the subset of the divisors of $f(m_2)$ also, and its intersection with the set common to the first two subsets. 1, -1 and 0 are the suggested values of m (provided they are not roots).

ELEMENTARY PROBLEMS AND SOLUTIONS

EDITED BY HOWARD EVES, Oregon State College

Send all communications concerning Elementary Problems and Solutions to Howard Eves, Mathematics Department, Oregon State College, Corvallis, Oregon. This department welcomes problems believed to be new, and demanding no tools beyond those ordinarily furnished in the first two years of college mathematics. To facilitate their consideration, solutions should be submitted on separate, signed sheets, within three months after publication of problems.

PROBLEMS FOR SOLUTION

E 950. *Proposed by W. R. Ransom, Tufts College*

Show that every positive integral power of $\sqrt{2}-1$ is of the form $\sqrt{m}-\sqrt{m-1}$.

E 951. *Proposed by C. S. Venkataraman, Trichur, South India*

Two circles are drawn touching the sides AB, AC of a triangle ABC at the ends of the base BC and also passing through the midpoint D of BC. If E is the other point of intersection of the circles show that AE is a symmedian of triangle ABC.

E 952. *Proposed by Albert Wilansky, Lehigh University*

Let $P:(r, \theta)$ be given in plane polar coordinates. Define the *p-slope* of the segment P_1P_2 to be $(r_2-r_1)/(\text{distance from } P_1 \text{ to } P_2)$. If A, B, C are three collinear points, with B between A and C, show that p-slope $CB \geqq p$-slope $CA \geqq p$-slope BA.

E 953. *Proposed by Joseph Langr, Prague, Czechoslovakia*

Four planes, drawn through a point, parallel to the faces of a tetrahedron, cut the edges of the tetrahedron in 12 points lying on a quadric surface.

E 954. *Proposed by G. W. Walker, Buffalo, N. Y.*

Let T_c be the cth positive integer which is both a triangular number and a square number. Find T_c as a function of c.

E 955. *Proposed by C. L. Dunsmore, University of California, Los Angeles*

Evaluate the nth order determinant $|a_{ij}|$, where $a_{ij}=1/(i+j-1)!$.

SOLUTIONS

Triangle, Circle, and Chords

E 915 [1950, 260]. *Proposed by Victor Thébault, Tennie, Sarthe, France*

The circle passing through the feet L, M, N of the cevians AL, BM, CN which divide the sides $BC=a$, $CA=b$, $AB=c$ of a triangle in the ratios

$$BL/LC = c^n/b^n, \quad CM/MA = a^n/c^n, \quad AN/NB = b^n/a^n,$$

intercepts on the lines BC, CA, AB signed segments of lengths x, y, z satisfying

the relation

$$x/a^{n-1} + y/b^{n-1} + z/c^{n-1} = 0.$$

Solution by the Proposer. Let LX, MY, NZ be the chords (or signed lengths x, y, z) intercepted by the circle LMN on the lines BC, CA, AB, and suppose the triangle lettered so that $a \geqq b \geqq c$. Now $AM \cdot AY = AN \cdot AZ$, with similar expressions favoring the other vertices. These relations yield the following equations:

$$bc^n[bc^n + (c^n + a^n)y]/(c^n + a^n)^2 = cb^n[cb^n + (a^n + b^n)z]/(a^n + b^n)^2,$$

$$ca^n[ca^n - (a^n + b^n)z]/(a^n + b^n)^2 = ac^n[ac^n + (b^n + c^n)x]/(b^n + c^n)^2,$$

$$ab^n[ab^n - (b^n + c^n)x]/(b^n + c^n)^2 = ba^n[ba^n - (c^n + a^n)y]/(c^n + a^n)^2.$$

Multiplying these equations, respectively, by a^{2n}, b^{2n}, c^{2n}, then adding and transposing all terms to the left, and finally dividing through by $a^n b^n c^n$, we obtain the required relation.

In $n=1$, in which case AL, BM, CN are the interior angle bisectors of the triangle, the relation reduces to

$$x + y + z = 0, \tag{1}$$

a result given by Lawrence in *The Mathematical Gazette*, 1945, p. 18.

If $n=2$, the relation becomes

$$x/a + y/b + z/c = 0. \tag{2}$$

The segments determined by the circle through the isotomic conjugates of L, M, N, satisfy the relation

$$a^{n+1}x + b^{n+1}y + c^{n+1}z = 0,$$

which reduces to (1) when $n=-1$ and to (2) when $n=-2$.

Also solved by Roger Lessard.

Editorial Note. This problem can lead to an interesting and extended study. Thus, suppose C_n and r_n are the center and radius of the circle LMN. What is the locus of C_n, and what is the expression for r_n in terms of a, b, c, n? Is there an extension of the tetrahedron? Etc.

Orthogonal Spheres

E 918 [1950, 335]. *Proposed by N. A. Court, University of Oklahoma*

Two spheres are orthogonal if they cut a given line in pairs of harmonic points and if the two planes determined by that line and the centers of the two spheres are perpendicular.

Solution by Roger Lessard, Ecole Polytechnique, Montreal. Let A_1, B_1 and A_2, B_2 be the intersections of the given line with the spheres S_1 and S_2 of centers O_1 and O_2 and radii r_1 and r_2. Let O and O' be the midpoints of A_1B_1 and A_2B_2.

Then O_1O and O_2O' are perpendicular to the given line. The harmonic relation gives

$$(OO')^2 = (OA_1)^2 + (O'A_2)^2,$$

whence the perpendicular relation gives

$$\begin{aligned}(O_1O_2)^2 &= (O_1O)^2 + (OO')^2 + (O'O_2)^2 \\ &= (O_1O)^2 + (OA_1)^2 + (O'A_2)^2 + (O'O_2)^2 \\ &= r_1^2 + r_2^2,\end{aligned}$$

and the spheres are orthogonal.

Also solved by the proposer.

An Expression for the Greatest Common Divisor

E 919 [1950, 335]. *Proposed by Leo Moser, University of Manitoba*

Show that the greatest common divisor of a and b is given by

$$(a, b) = \sum_{m=0}^{a-1} \sum_{n=0}^{a-1} (1/a)e^{2\pi i b mn/a}.$$

Solution by F. Bagemihl and W. Seidel, University of Rochester. Let $d = (a, b)$. Then $a = rd$, $b = sd$, where $(r, s) = 1$. Also, $(b/a)m = (s/r)m$, which is an integer for precisely d values of $m \leqq a-1$, namely, $m = 0, r, 2r, \cdots, (d-1)r$. Now if $(b/a)m$ is not an integer

$$\sum_{n=0}^{a-1} e^{2\pi i b mn/a}$$

is a finite geometric series with sum zero. But if $(b/a)m$ is an integer, the same sum is equal to a, since there are then a terms each equal to 1. Hence

$$\sum_{m=0}^{a-1} \sum_{n=0}^{a-1} (1/a)e^{2\pi i b mn/a} = (1/a)da = d.$$

Also solved by P. M. Anselone and V. E. Hoggatt (jointly), D. H. Browne, N. J. Fine, Roger Lessard, A. E. Livingston, H. Ohbayashi, L. B. Rall, L. A. Ringenberg, C. M. Sandwick, Sr., P. J. Schillo, N. T. Seely, Jr., and the proposer.

The proposer pointed out that the expression for (a, b) is given without proof in G. Frobenius, *Vorlesungen über Zahlentheorie*, p. 28. The well known properties of (a, b), such as $(a, b) = (b, a)$ and $(ca, cb) = c(a, b)$, may easily be deduced from it.

A Matrix Identity

E 920 [1950, 335]. *Proposed by T. G. Room, University of Sydney, Australia*

If S and T are any two square matrices of the same order, and the necessary matrices are non-singular, then

$$(I+S)^{-1}(S+T)(I+ST)^{-1}(I+S) = (I-S)(I+TS)^{-1}(S+T)(I-S)^{-1}.$$

Solution by the Proposer. Set

$$A = (S+T)(I+ST)^{-1}(I-S^2),$$

$$B = (I-S^2)(I+TS)^{-1}(S+T).$$

We have to prove that $A=B$. Now we clearly have

$$T(I+ST)^{-1} = (I+TS)^{-1}T, \tag{1}$$

$$T(I+ST)^{-1}S = I-(I+TS)^{-1}, \tag{2}$$

$$(I+ST)^{-1}S = S(I+TS)^{-1}, \tag{3}$$

$$(I+ST)^{-1} = I-S(I+TS)^{-1}T. \tag{4}$$

(The proofs of (1) and (2) are immediate, and (3) and (4) are the same results with S and T interchanged.) Using these four results we now have

$$\begin{aligned} A &= S(I+ST)^{-1} + T(I+ST)^{-1} - S(I+ST)^{-1}S^2 - T(I+ST)^{-1}S^2 \\ &= S - S^2(I+TS)^{-1}T + (I+TS)^{-1}T - S^2(I+TS)^{-1}S \\ &\quad - S + (I+TS)^{-1}S = B. \end{aligned}$$

Also solved by Mohammad Ishaq and Ingram Olkin.

A Multiple-pan Balance

E 921 [1950, 416]. *Proposed by R. T. Hood, Beloit College*

Devise a multiple-pan balance, with minimum number of pans, by which one can weigh any whole number of pounds using weights which are powers of n, n a given positive integer. What other amounts can be weighed with such a balance?

Solution by S. T. Thompson and the Proposer. The required balance has $n-2$ pans on the right side of the fulcrum and one on the left, such that their points of suspension and the fulcrum constitute n equally spaced points. Let the pans be numbered from left to right by $-1, 1, 2, 3, \cdots, n-2$. Suppose p pounds are to be weighed. Let the sequence of digits representing p in the n-ary system be altered by starting from the right and replacing each digit, except 1, by that digit minus n, and at the same time increasing the next digit to the left by 1. Then if the new jth digit is k, put the weight of n^{j-1} pounds in the pan numbered $-k$. This arrangement of weights will balance p/q pounds placed in pan q, $q=1, 2, \cdots, n-2$.

Editorial Note. The above is a generalization of the well known fact that an ordinary two-pan balance can weigh any whole number of pounds by using weights of 1, 3, 3^2, $\cdots$ pounds, if weights are permitted in both pans. The crux of the solution rests on the fact that every positive integer may be represented as a series of multiples by $1, 0, -1, \cdots, -(n-2)$ of the successive powers of n.

A Dissection of a Triangle

E 922 [1950, 416]. *Proposed by Michael Goldberg, Washington, D. C.*

Show that any given triangle can be dissected by straight cuts into four pieces which can be arranged to form two triangles similar to the given triangle.

Solution by Aaron Buchman, Buffalo, New York. Take D, E, F on AB, BC, CA, the sides of any triangle ABC, such that $AD/AB = CE/CB = AF/AC = 1/5$. Let G be on AC with $AG = 2AF$, and let H be the midpoint of DE. Then the segments DE, DF, GH represent the three cuts. One of the required triangles is the piece BDE. The other three pieces may be rotated in the plane to form the second triangle. The proof is elementary.

Also solved by D. H. Browne, Joseph Langr, Roger Lessard, B. D. Roberts, Joseph Rosenbaum, Harold Shniad, C. W. Trigg, and the proposer.

A great variety of solutions was offered for the problem, but most of them, like the above, led to dissections where the linear ratios of the three triangles in question are 3:4:5. In some of the solutions certain pieces had to be turned over. Remarkable ingenuity was exercised by Rosenbaum, who offered a surprising number of solutions. In addition to many of the 3:4:5 type he found dissections where the linear ratios of the three triangles are, for example,

$$(a_1 + a_2)^2 - a_i^2 : 2(a_1 + a_2)a_i : (a_1 + a_2)^2 + a_i^2, \qquad i = 1, 2, 3,$$

where a_1, $a_2 \leqq a_1$, a_3 are the sides of the given triangle. He also found solutions applicable to certain limited classes of triangles. In particular, he found a ***three*** piece solution applicable to any nonequilateral triangle. Here the ratios are

$$2a_1a_2 : a_1^2 - a_2^2 : a_1^2 + a_2^2, \qquad a_1 > a_2.$$

One wonders if there is a three piece solution for the equilateral triangle.

ADVANCED PROBLEMS AND SOLUTIONS

EDITED BY E. P. STARKE, Rutgers University

Send all communications concerning Advanced Problems and Solutions to E. P. Starke, Rutgers University, New Brunswick, New Jersey. All manuscripts should be typewritten, with double spacing and with margins at least one inch wide. Problems containing results believed to be new or extensions of old results are especially sought. Proposers of problems should also enclose any solutions or information that will assist the editor. In general, problems in well known text books or results found in readily accessible sources should not be proposed for this department.

PROBLEMS FOR SOLUTION

4425. *Proposed by P. A. Piza, San Juan, Puerto Rico*

There are infinitely many triangular numbers (numbers of the form $k(k+1)/2$) which are also perfect squares, viz.

$$[(17 + 12\sqrt{2})^n + (17 - 12\sqrt{2})^n - 2]/32.$$

Find numbers which are at the same time a sum of two squares and a sum of two triangular numbers.

4426. *Proposed by D. J. Newman, New York University*

Let a, b, c, d all have modulus unity. Prove that in the unit circle the polynomial

$$P(z) = az^3 + bz^2 + cz + d$$

has a maximum modulus not less than $\sqrt{6}$. What is the result for a polynomial of degree greater than three?

4427. *Proposed by Paul Erdös, University of Aberdeen, Scotland*

Let

$$f_n(x) = \prod_{i=1}^{n} (x - x_i), \qquad -1 \leqq x_i \leqq 1.$$

Prove that there cannot exist numbers a, b such that

$$|f_n(a)| \geqq 1, \qquad |f_n(b)| \geqq 1, \qquad -1 < a < 0 < b < 1.$$

SOLUTIONS

Square Inscribed in Arbitrary Simple Closed Curve

4325 [1949, 39]. *Proposed by Orrin Frink, Pennsylvania State College.*

Show that on every simple closed plane curve there are four points which are the vertices of a square.

Note by A. M. Gleason, Cambridge, Massachusetts. I would like to point out

the inadequacy of the proof printed in the June issue [1950, 423] as a solution of problem 4325. First, it is stated that it is possible to deform a horizontal chord in such a way that it is always horizontal and moves from the first contact to the last contact. While this is easy to see when the curve has only a finite number of maxima and minima, it certainly requires proof that it can be done in general.

Second, the perpendicular bisector of the moving horizontal chord is alleged to have continuous behavior. Since the perpendicular bisector of the curve may have many different intersections with the curve it is easy to give examples in which, say, the first intersection counting from below is not continuous. It may be possible to get around this difficulty by the device of retrograding as before, but this must be combined with the demand for a retrograde motion for XX' and proof is certainly required.

Third, it is assumed that the rhombi constructed for each direction depend continuously on the direction. This is a difficulty which cannot be circumvented by the devices previously employed. Let A, B, C, D be the vertices of a rhombus. A simple closed curve may be easily constructed containing A, B, C, D but not containing any other rhombus near $ABCD$. To do this, simply let A and C lie on cusps pointing to the right while B and D lie on cusps pointing to the left.

Note by J. J. Schaeffer, Instituto de Matematica y Estadistica, Montevideo, Uruguay. In the second part of the proof the greatest difficulties are encountered. This applies particularly to the statement "then rotate m_x continuously through 90° until it reaches the position m_y; now $XX' < YY'$." There seems to be no reason to assume the possibility of returning by continuous rotation to the initial position of the rhombus, and even in the case of fairly simple curves the proof encounters apparently insurmountable obstacles.

For convex figures, however, the idea can be used, as it can be easily proved that convex figures have the property that for any slope there is essentially only one rhombus with a diagonal parallel to this slope. This is the same idea as Emch's, but in order to apply it to general convex curves, which may contain rectilinear segments or angles, the use of continuity should be avoided, and replaced by considerations of limit points of open sets.

Similar comments were received from Arthur Rosenthal, and G. A. Dirac. Further information and discussion of this interesting problem will be welcomed.

A Prime Representing Function

4337 [1949, 186]. *Proposed unsigned.*

If the numbers R_{nk} are defined by

$$\frac{1 - z^2}{\sin \pi z} \prod_{k=2}^{n} \sin (\pi z/k) \equiv \sum_{k=0}^{\infty} R_{nk} z^k,$$

prove that $\underline{\lim}\, R_{nk}^{-1/k}$ is equal to the first prime exceeding n.

Solution by the Proposer. The only possible poles are at the integers, and these are cancelled by zeros of the numerator for all z such that $|z| \leqq n$. As $|z|$ increases beyond n, the numerator will first fail to vanish when z is a prime. The radius of convergence, then, is equal to the distance to the first prime beyond n, and the desired result follows immediately.

Two Feuerbach Point Theorems

4344 [1949, 269]. *Proposed by Victor Thébault, Tennie, Sarthe, France*

(1) If, in a triangle, one of the angles is 120° (or 60°), two of the Feuerbach points are diametrically opposite on the nine-point circle, and conversely. (2) If the triangle is scalene and if the circle through the feet of the interior bisectors (or one interior and two exterior bisectors) passes through one of the vertices, three of the Feuerbach points form an isosceles triangle, and conversely.

Solution by J. W. Clawson, Ursinus College, Collegeville, Pennsylvania. Let ABC be the triangle, and A', B', C' the midpoints of the sides. We shall use the triangle $A'B'C'$ as basic and employ trilinear areal coördinates, taking $B'C' = a$, and so on.

The equation of the nine-point circle of ABC is

$$a^2yz + b^2zx + c^2xy = 0;$$

that of the inscribed circle is

$$a^2yz + b^2zx + c^2xy = [(b-c)^2x + (c-a)^2y + (a-b)^2z](x+y+z);$$

and that of the escribed circle opposite A is

$$a^2yz + b^2zx + c^2xy = [(b-c)^2x + (c+a)^2y + (a+b)^2z](x+y+z).$$

Then the Feuerbach point, F, at which the nine-point circle and the inscribed circle touch has coördinates

$$\frac{a}{b-c}, \quad \frac{b}{c-a}, \quad \frac{c}{a-b};$$

while F_1, at which the nine-point circle touches the above escribed circle is given by

$$\frac{a}{b-c}, \quad \frac{-b}{c+a}, \quad \frac{c}{a+b}.$$

Similarly the other two Feuerbach points, F_2, F_3, are respectively

$$\frac{a}{b+c}, \quad \frac{b}{c-a}, \quad \frac{-c}{a+b}; \qquad \frac{-a}{b+c}, \quad \frac{b}{c+a}, \quad \frac{c}{a-b}.$$

The center of the nine-point circle, N, is

$$a^2(b^2+c^2-a^2), \qquad b^2(c^2+a^2-b^2), \qquad c^2(a^2+b^2-c^2).$$

(1) The condition that F, F_1, N are collinear is easily reduced to $a^2=b^2+c^2-bc$, which means that $A'=A=60°$. The condition that F_2, F_3, N are collinear is $a^2=b^2+c^2+bc$, which means that $A'=A=120°$.

(2) The point P at which the interior bisector of angle A meets BC is $b+c$, $c-b$, $b-c$; P', the foot of the external bisector of A, is $b-c$, $-b-c$, $b+c$; with analogous expressions for Q, Q', R, R', the feet of the other bisectors. By substituting in the general equation of a circle

$$a^2yz+b^2zx+c^2xy=(px+qy+rz)(x+y+z),$$

and eliminating p, q, r, one finds the condition that P, Q, R, A be concyclic is

$$a^3+a^2(b+c)-a(b^2+bc+c^2)-(b^3+b^2c+bc^2+c^3)=0.$$

The same equation is obtained, somewhat laboriously, as the condition that $F_1F_2=F_1F_3$. We note that $a=b$ reduces the above equation to $c(b+c)^2=0$. Hence if the triangle is not scalene, this condition could not be satisfied for all three of its cases.

In the same way we may deduce that the condition for P, Q', R', A to be concyclic is the same as that for $FF_2=FF_3$.

Also solved by R. Goormaghtigh, and the proposer.

Editorial Note. Goormaghtigh derives the solution synthetically from the property: when a circumdiameter of a triangle ABC moves about the circumcenter, its orthopole as to ABC moves on the nine-point circle with an angular speed twice that of the circumdiameter. He gives also the following references:

Part (1) has been mentioned before. See Goormaghtigh, *Tôhoku Mathematical Journal*, 1926, p. 107.

For properties of triangles having one angle equal to 60° or 120°, see Neuberg, *Mathesis*, 1895, p. 215, Droz-Farny, Deprez, *Ibid.*, 1897, p. 77, Goormaghtigh, *Ibid.*, 1914, p. 107.

For properties of the special triangles considered in part (2), see Cristeseu, *Gazeta Matematica*, 1902–1903, p. 112, Bastin, *Mathesis*, 1903, p. 32, Neuberg, *Ibid.*, 1903, p. 134.

Properties of these and other special triangles will be found in Neuberg, Bibliographie des triangles spéciaux, *Mémoires de la Société royale des Sciences de Liége*, 1924, Goormaghtigh, Propriétés nouvelles des triangles spéciaux, *Ibid.*, 1925.

Euler's Constant

4353 [1949, 479]. *Proposed by H. F. Sandham, Trinity College, Ireland*

Prove that

$$\sum_{n=1}^{\infty} \frac{(-1)^n}{n}\left[\frac{\log n}{\log 2}\right] = \gamma,$$

where $[x]$ denotes the integral part of x, and γ is Euler's constant.

Solution by D. F. Barrow, University of Georgia. From the definition of γ if follows that

$$\gamma = \lim_{n\to\infty}\left(\sum_{s=1}^{2^n} \frac{1}{s} - \log 2^n\right).$$

The expression in parentheses may be written

$$\sum_{s=1}^{2^n} \frac{1}{s} - \log 2 - \log 2^{n-1} = \sum_{s=1}^{2^n} \frac{1}{s} + \sum_{s=1}^{\infty} \frac{(-1)^s}{s} - \log 2^{n-1}.$$

Now, when the terms of the finite series are added to the corresponding terms of the infinite series, the odd terms cancel and the even terms add. Thus we have

$$\sum_{s=1}^{2^{n-1}} \frac{1}{s} - \log 2^{n-1} + \sum_{s=2^n+1}^{\infty} \frac{(-1)^s}{s}.$$

Next apply the same reduction to the first two terms of this expression, obtaining

$$\sum_{s=1}^{2^{n-2}} \frac{1}{s} - \log 2^{n-2} + \sum_{s=2^{n-1}+1}^{\infty} \frac{(-1)^s}{s} + \sum_{s=2^n+1}^{\infty} \frac{(-1)^s}{s}.$$

Let this process be continued until the logarithmic term is used up, and then, in the various series that arise, let terms having a common denominator be added:

$$1 + \sum_{r=1}^{n-1}\left(r \sum_{s=2^r+1}^{2^{r+1}} \frac{(-1)^s}{s}\right) + n \sum_{s=2^n+1}^{\infty} \frac{(-1)^s}{s}.$$

As n grows infinite, we notice that the last term, containing the infinite limit, approaches zero, and this leads to

$$\gamma = 1 + \sum_{s=1}^{\infty} \frac{(-1)^s}{s}\left\{\frac{\log s}{\log 2}\right\},$$

where $\{x\}$ denotes the largest integer less than x. This series becomes the one called for in the problem if we express the initial 1 as the infinite series $1/2+1/4+1/8+\cdots$, and add these terms to those of the infinite summation having like denominators.

Also solved by M. S. Klamkin, Norman Miller, and the proposer.

RECENT PUBLICATIONS

EDITED BY E. P. VANCE, Oberlin College

All books for review should be sent directly to the editor of this department, American Mathematical Monthly, 531 West 116th Street, New York 27, N. Y. and not to any of the other editors or officers of the Association.

University Mathematics. By Joseph Blakley. Glasgow, Blackie & Son, Limited, 1949. 527 pages. 25s.

This is a University textbook for students of science and engineering and is intended to be used in a first-year course in Pure Mathematics. It is essentially a book on analytic geometry and calculus but contains as well two chapters on differential equations including a section on partial differential equations.

The material covered is well integrated and due emphasis is placed on matters of manipulation and drill but not enough attention is paid to questions of theory and rigor. "The limit of the quotient of two functions is equal to the quotient of their limits" (p. 14). In the presentation of partial fractions the statement is made that for every cubic factor ax^3+bx^2+cx+d there will be a term $(Ax^2+Bx+C)/(ax^3+bx^2+cx+d)$ and, further, it is noted that "a quadratic factor or a cubic factor cannot be factorized further" (p. 70). A power series is defined in terms of coefficients which are supposed to be rational (p. 165). Under partial differentiation it is stated that "if $u=f(x, y)$ be a continuous function, then $\partial^2u/\partial x\partial y=\partial^2u/\partial y\partial x$" (p. 210).

By way of explaining the topics under discussion the author works many illustrative problems and there is a set of examples (the only ones given) at the end of each chapter with answers supplied at the back of the book. These examples—more than 500 in number—are mostly taken from London University examination papers and hence would not in general be readily adaptable to use as daily assignments in a regulation American College course.

C. O. OAKLEY

The Mathematics of Great Amateurs. By J. L. Coolidge. Oxford University Press, 1949. 8+211 pages. $6.00.

In his preface the author admits difficulty in defining an amateur in mathematics, and additional difficulty in choosing a limited few from the many who conform to his definition. We would enter upon no debate with him upon his choice in these matters. Intriguing indeed are most of the names which he has chosen—one for each of his sixteen chapters—and, likewise, the topics under each name. They deserve listing here.

Plato starts the list. We have mention of his mathematical training, followed by various topics and ending with his thoughts on commensurability of numbers. Next is Omar Khayyám, his story and his contribution to cubics. Then comes Leonardo, his background and, naturally, a variety of topics. We then have Dürer with his study of spirals and helices and his lead into descrip-

tive geometry through study of measurements of the human body; Napier, logarithms and various other topics, followed by a word of deserved praise; Pascal, his famous theorem, logic of mathematics and other discussions, including integration; Arnauld who, probably, made the first serious attempt to break away from Euclid, with his logic of mathematics and his new approach to geometry; De Witt, his writings on geometry and on annuities; Hudde, his work on reduction of equations and on equal roots; Brouncker on continued fractions and other studies; L'Hospital, several contributions ending with infinitesimals and conics. Then come Buffon, his search for the infinite, moral arithmetic, probability; Diderot on vibrating strings, involutes, the pendulum; Horner and his famous "method"; and, finally, Bolzano with his study of real functions and his close approach to Cantor's idea of the power of an infinite assemblage.

The very nature of such a work invites—and the author accepts the invitation—the idea of making the treatment replete with the human element. One feels that he is delightfully successful in this. It is refreshing to go back with the author and to rethink the thoughts of those who worked when, mathematically speaking, the "world was young." It is to be hoped that his wish that someone might undertake the task of continuing this type of study and writing, might be granted.

F. B. WILEY

Regular Polytopes. By H. S. M. Coxeter. London, Methuen, 1948; New York, Pitman, 1949. xx+321 pages. $10.00.

A polytope (to give an informal definition) is a figure bounded by line segments, or by portions of planes or hyperplanes; the word is used to describe any member of the sequence *point, segment, polygon, polyhedron, etc.* The study of polytopes in their simplest forms goes back at least as far as the Greeks and their investigations of the properties of polygons and polyhedra. However the middle of the nineteenth century had arrived before Schläfli, one of the first to venture into space of more than three dimensions, discovered the general polytope. His work attracted little attention, and it was not until 1880 that his ideas were rediscovered by Stringham and again later by others. With the publication of this book further rediscoveries are not to be expected.

Professor Coxeter has collected and arranged the scattered material, enriched it with the results of his own research done over a period of twenty-five years, and has presented us with what is, and will probably be for many years, the only organized treatment of the subject. For this reason we should be glad that the work has been done by such a competent scholar and by one who is doubtless more thoroughly acquainted with this field than anyone else.

The arrangement of the material is excellent. The first six chapters are devoted largely to the theory of polyhedra, leading the reader by easy stages from the familiar five Platonic solids, through a mild change in the definition of polyhedron, to the four beautiful Kepler-Poinsot star-polyhedra, which are also

regular. These chapters serve at the same time to introduce methods and concepts which later prove effective in the study of polytopes in higher space. Among these are the use of maps, symmetry groups, honeycombs, truncation, stellating, faceting, and the author's own ingenious *graphical symbolism*. The remaining eight chapters extend the pattern to the hypersolids, and here are encountered the regular polytopes, of which there are sixteen in four-space and three in each of the other higher dimensional spaces.

Nothing of much importance appears to have been omitted. One may regret the exclusion of any discussion of regular skew polyhedra, but apparently there was not room for that. The historical notes, of which there is a rich collection, constitute one of the most fascinating parts of the book. In addition to numerous excellent drawings, there is a set of eight plates furnished by P. S. Donchian, showing various solids and some remarkable models which are solid projections of regular four-dimensional polytopes. Many readers will recall seeing these models at the Chicago Exposition.

The theory of polytopes has attracted the interest of numerous amateur mathematicians, a number of whom have been quite gifted and have made valuable contributions. It is to be hoped that this book will introduce to the subject others who study mathematics for the love of it. From this group there may emerge one of those rare individuals possessing the mysterious power to visualize the hypersolids as easily as most of us visualize the solids of three-space. And there is still work for these amateurs to do. For example, the skew polytopes have not been investigated at all. They will not find it easy, at least after the first few stages, but this will not deter them. One recalls that difficulties have not discouraged amateurs in the study of number theory.

This book belongs in every university, college and high school library, and on the bookshelf of every mathematician. It is a book to read, to refer to and to recommend to students.

H. E. WOLFE

NEW BOOKS RECEIVED

First Course in Probability and Statistics. By J. Neyman. New York, Henry Holt, 1950. 10+350 pp. $3.50.

Practical Mathematics. By C. Palmer and S. Bibb. Part 2, Algebra with Applications, 5th Edition. New York, McGraw-Hill, 1950. 14+252 pp. $2.20.

A Theory of Formal Deducibility. (Notre Dame Mathematical Lectures, No. 6.) By H. Curry. Notre Dame, Indiana, 1950. 10+126 pp.

Theorie des Distributions, Part i. (L'Université de Strasbourg, Vol. 9). By L. Schwartz. Paris, Hermann, 1950. 148 pp.

College Mathematics. By C. Clark. New York, Prentice-Hall, 1950. 6+331 +46 pp. tables. $3.85.

Analytic Geometry. By R. Douglass and S. Zeldin. New York, McGraw-Hill, 1950. 10+216 pp. $2.75.

Contributions to Mathematical Statistics. By R. A. Fisher. New York, Wiley, 1950. 656 pp. $7.50.

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Mathematics of Relativity. By G. Y. Rainich. New York, John Wiley and Sons, Inc., 1950. vii+173 pp. $3.50.

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Business Mathematics, 3rd Edition. By C. Richtmeyer and J. Foust. New York, McGraw-Hill, 1950. 18+441 pp. $3.50.

Leçons sur les nombres transfinis. By W. Sierpinski. Paris, Gauthier-Villars, 1950. 7+240 pp.

Algebra: Its Big Ideas and Basic Skills. By Aiken Henderson. New York, Harper and Brothers, Publishers, 1950. xv+409 pp.

A Course in the Slide Rule and Logarithms. Revised Edition. By E. J. Hills. Boston, Ginn and Company, 1950. iv+108 pp. $1.40.

Differential Equations. By J. E. Powell and C. P. Wells. Boston, Ginn and Company, 1950. vi+205 pp. $3.00.

Calculus and Analytic Geometry. By C. T. Holmes. New York, McGraw-Hill, 1950. x+416 pp. $4.75.

Fundamentals of the Calculus. By D. E. Richmond. New York, McGraw-Hill, 1950. xi+233 pp. $3.00.

Ordinary Differential Equations. By Michael Golomb and Merrill Shanks. New York, McGraw-Hill, 1950. ix+356 pp. $3.50.

Royal Society of Mathematical Tables I, The Farey Series of Order 1025. By E. H. Neville. New York, Cambridge University Press, 1950. xxvii+2 pp. Decimal Index + 405 pp. $18.50.

Numerical Mathematical Analysis. Second Edition. By J. B. Scarborough. Baltimore, The Johns Hopkins Press, 1950. xviii+511 pp. $6.00.

Bücher der Mathematik und Naturwissenschaften, Differential- und Integralrechnung. By Wilhelm Maak. Wolfenbuttel and Hannover, Wolfenbutteler Verlagsanstalt, 1949. 235 pp.

Riemannian Geometry. By L. P. Eisenhart. Princeton, Princeton University Press, 1949. vii+306 pp.

Vorlesungen Über Analytische Geometrie des Raumes. By Karl Kommerell. Stuttgart, K. F. Koehler Verlag. viii+407 pp. DM 26.

Fourier Series. Translation of Rogosinski's original (by Samml. Goschen, 1930) by Harvey Cohn and F. Steinhardt. New York, Chelsea Publishing Co., 1950. vi+176 pp.

Grundbegriffe Moderner Statistischer Methodik, Erster Teil. By L. von Baranow. Stuttgart, S. Hirzel Verlag, 1950. viii, 112 Seiten mit 16 Abb. DM 6.

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Theory of Mental Tests. By Harold Gulliksen. New York, John Wiley and Sons, Inc., 1950. xix+486 pp. $6.00.

Theory of the Interior Ballistics of Guns. By J. Corner. New York, John Wiley and Sons, 1950. xiii+443 pp. $8.00.

Mathematical Snapshots. By H. Steinhaus. New York, Oxford University Press, 1950. vi+266 pp. $4.50.

Formulas and Theorems for the Special Functions of Mathematical Physics. By Wilhelm Magnus and Fritz Oberhettinger. New York. Chelsea Publishing Company, 1949. 172 pp.

Mecanique des Milieux Continus et Deformables, Tome I, Parts 1 and 2. By Maurice Roy. (Part I: Thermodynamique et mecanique des milieus continus et deformables. Part II: Equilibre et Mouvement des solides elastiques) Paris, Gauthier-Villars, 1950. xxii+366 pp. $8.36.

Mecanique des Milieux Continus et Deformables, Tome II, Parts 3 and 4. By Maurice Roy. (Part III: Equilibre et Mouvement des Fluides. Part IV: Theorie des Machines) Paris, Gauthier-Villars, 1950. xii+338 pp. $6.87.

Electromagnetic Waves. By F. W. G. White. New York, John Wiley and Sons, 1950. 108 pp. $1.25. (Methuen's Monographs on Physical Subjects).

The Special Theory of Relativity. By Herbert Dingle. (Methuen's Monographs on Physical Subjects). New York, John Wiley and Sons, 1950. 94 pp. $1.25.

Wave Guides. By H. R. L. Lamong. New York, John Wiley and Sons, 1950. 117 pp. $1.50.

Relativity Physics. By W. H. McCrea. New York, John Wiley and Sons, 1950. 87 pp. $1.25.

Bertrand Russell, A Sketch of His Life and Work. By H. W. Leggett. New York, Philosophical Library, 1950. 78 pp. $3.75.

The Philosophy of Mathematics. By E. A. Maziarz. New York, Philosophical Library, 1950. viii+286 pp. $4.00.

Strategy in Poker, Business and War. By John McDonald. New York, W. W. Norton and Company, Inc., 1950. 128 pp. $2.50.

Grundzuge der Galois'schen Theorie (*N. Tschebotarow*). Translated and revised by H. Schwerdtfeger. Holland, P. Noordhoff, 1950. v+432 pp. f 17.50, geb f 20.

Irrational and Rational Number Theory. By E. Schlytter. Chicago, 1950. Copyright by author. 67 pp. +ii index.

Weltsystem Weltäther und die Relativitätstheorie. By Karl Jellinke. Switzerland, Wepf and Co., 1949. xv+450 pp. SFr. 45.

Verstandliche Elemente der Wellenmechanik, I Teil. By Karl Jellinek. Switzerland, Wepf and Co., 1950. xii+304 pp. SFr. 34.

Ausgleichsrechnung. By V. Happach. Leipzig, Teubner, 1950. 104 pp. $1.50.

CLUBS AND ALLIED ACTIVITIES

EDITED BY L. F. OLLMANN, Hofstra College

Send reports of all activities, such as club reports, special features, topics with references, student papers, and other material of interest to L. F. Ollmann, Hofstra College, Hempstead, New York.

CLUB REPORTS, 1949–50

Pi Mu Epsilon, University of Pennsylvania

The *Pi Mu Epsilon* chapter at the University of Pennsylvania held regular meetings every month. The papers presented were:

Parallel curves, by Prof. C. B. Allendoerfer

Rational and irrational numbers, by Prof. Brinkman

Sylvester's identity, by Herman Zabronsky

P-adic numbers, by Dr. Emil Grosswald

The role of electronic computers in mathematics, by Dr. Herbert Mitchell

Logic of the circuits of electronic digital computers, by Prof. G. W. Patterson.

The activities for the year closed with a dinner at which Prof. J. R. Kline was the featured speaker.

Officers for the year 1950–51 are: Director, Dr. R. D. Anderson; President, Herbert Gurk; Secretary, David Loev; Treasurer, Mildred Goss.

Mathematics Club, Butler University

The *Mathematics Club* of Butler University held regular meetings twice a month during the academic year. At these meetings a paper is presented by a student, faculty member, or guest.

The club also sponsors two outings and a Christmas party each year. One of the outings was a trip to the Johnson's Observatory, while the other two were a Christmas party and a picnic.

Titles and authors of papers presented at the regular meetings are:

The geometric fourth dimension, by Prof. H. E. Crull

Nomography, by W. W. Gorsline

Solids, regular and semi-regular, by Carol Wilson

Isaac Newton, by Eldon Palmer

Applications of mathematics to mechanics, by Kenneth Hallam

Application of mathematics to submarine detection and torpedo firing data, by Dr. H. E. Crull

Geometrical paradoxes, by Miss Jane Uhrhan

The duo-decimal system, by Edwin Britz

Dynamic symmetry in Greek art, by Prof. June Beal

Mathematical paradoxes, by Egon Steinkamp

Several proofs of the Pythagorean theorem, by Frank Rexroth

The binary system, by Ted Souders.

Mathematics Club, University of Kansas

The *Mathematics Club* of the University of Kansas presented the following programs during 1949–50:

Algebraic numbers, by Dr. Robert Schatten
Is the sum of algebraic numbers an algebraic number?, by Robert Fisher
Methods of trisecting any angle, by Dr. G. W. Smith
Unsolved problems in number theory, by Dr. S. Chowla
The proof of the irrationality of pi, by Dr. I. N. Herstein
The planimeter, by Ralph Simmons
A problem in statistics, by Dr. G. B. Price
Schwartz's inequality, by Mr. Kuo-Chih Hsu
Elementary cryptography, by Dr. G. W. Smith
Combinatorial topology, by Calvin Foreman
Reflected triangles, by Arthur Kruse.

The annual picnic was held in the spring. Officers for the year 1950–51 are: President, Robert Newton; Vice-President, Marion Brown; Secretary-Treasurer, Ruth Hurwitz.

Pi Mu Epsilon, Brooklyn College

The following talks were given at open meetings of the *New York Gamma* chapter of *Pi Mu Epsilon* during the academic year 1949–50:

The concept of homotopy, by Prof. Samuel Eilenberg of Columbia University
Topology, by Prof. Norman Steenrod of Princeton University
Simple Simon, miniature mechanical brain, a lecture-demonstration by Edmund C. Berkeley, a consultant mathematician and author of the book "Giant Brains."

Two installations of new members were held during the year. On both occasions, refreshments were served, and a social event followed the initiation ceremonies.

Officers for the Fall term, 1950–51 are: President, Anotole Beck; Vice-President, Lawrence Bennett; Financial Secretary, George Booth; Correspondence Secretary, Alvin Hausner; Director, Prof. Moses Richardson.

Kappa Mu Epsilon, Iowa State Teachers College

The following papers were presented to the *Iowa Alpha* chapter of *Kappa Mu Epsilon*:

Paper constructions, by Lena Abbas
Eisenstein's irreducible criteria, by Don Edwards
Magic squares, by Sam Weigert
The theory and uses of the planimeter, by Don Richardson
Problem of Apollonius, by Eddie Sage
Boolean algebra, by Robert Robinson
Inversions, by Marian Karrys.

The Homecoming Breakfast was held in the Fall at the home of Dr. H. Van Engen, Sponsor of the chapter and National President of *Kappa Mu Epsilon.*

Nine student members attended the convention in Chicago during April, 1950, at which time Robert Allender presented a paper to the gathering.

Officers for 1950–51 are: President, George York; Vice-President, Sam Weigert; Secretary-Treasurer, Gladys Satell; Corresponding Secretary, Prof. George Keppers; Sponsor, Dr. H. Van Engen.

Sigma Phi Mu, Montclair Teachers College

This year's program began with the initiation and welcoming of the Freshmen into *Sigma Phi Mu.* The social program consisted of an annual Christmas party with the *New Jersey Beta* chapter of *Kappa Mu Epsilon,* an interesting and pleasant Probability Night and an annual May picnic.

Papers presented during 1949–50 were:

The use of the mathematics laboratory, by George Kays

The conic sections, by Charles Sensale

So you want to be a mathematics teacher!, by Ralph C. Miller

The functionalism of teaching mathematics in the high school, by Dr. V. S. Mallory

The trisection of an angle by means of various curves, by Audrey Jensen

Interesting facts about e, by William Koellner

Teaching mathematics in the elementary schools, by Lawrence Campbell.

The officers for the coming year are: President, Marie Kanthack; Vice-President, Ruth Maehl; Secretary, Margaret Cotter; Treasurer, Phyllis Friedman.

Kappa Mu Epsilon, Washburn Municipal University

At the monthly meetings of the *Kansas Delta* chapter of *Kappa Mu Epsilon,* the following papers were given:

Theory of numbers, by Dr. S. Chowla, guest speaker

Problems in algebra, a panel discussion, by Hugo Rolfs, Howard Sperry, William Powell, and Kenneth Lake

The trisection of an angle, a panel discussion, by Norman Hoover, Margery Gamble, Edna Metzenthin and Nancy Martin

The mathematical needs of the psychology major, by Robert Hage of the Department of Psychology

On the Hausdorff summability of derived and conjugate derived Fourier series, by Dr. P. E. Eberhart

Zeno's paradoxes, by Kenneth Lake

A few of the interesting topics in algebra, by Betty Jane Moffett

Mobius rings, by Dale Long.

One of the most successful meetings of the year was High School Senior Guest Night. Approximately seventy persons attended the meeting, which consisted of greetings from the President of the University and from the Chairman

of the Department of Mathematics. Four student members of *Kappa Mu Epsilon* gave short talks on mathematics, and the President of the chapter explained *Kappa Mu Epsilon* and its project. We had an enjoyable informal refreshment hour, after which we went to the observatory to look through the telescope. Many notes and calls of appreciation were received from the parents and the seniors from our four surrounding high schools.

Dr. S. Chowla, of India, who is a visiting professor at the University of Kansas, and a former participant in the Institute for Advanced Study, was the guest speaker for the initiation dinner in the fall. Dr. G. Baley Price, an honorary member of the chapter, was also a guest.

Miss Laura Greene, *Kansas Delta* Sponsor, was in charge of the arrangements for the *Kappa Mu Epsilon* luncheon held in connection with the National Council of Teachers of Mathematics meeting in Wichita, Kansas. Dr. O. J. Peterson, *Kansas Beta* chapter of *Kappa Mu Epsilon*, presided at the meeting. Kenneth Lake, spoke on the topic *An elementary discussion of fundamental concepts in modern algebra*. Approximately twenty-five persons attended the luncheon at which Dr. E. H. C. Hildebrandt, honorary member of *New York Alpha*, was a guest.

The officers for 1950–51 are: President, Joe Latas; Vice-President, Dale Long; Secretary, Narra Smith; Treasurer, Margery Gamble; Sponsor, Miss Laura Greene; Corresponding Secretary, Miss Margaret Martinson.

Kappa Mu Epsilon, Central Michigan College of Education

Reports read to the *Michigan Beta* chapter of *Kappa Mu Epsilon* during 1949–50 were:

Mathematics in music, by Margaret King
Mathematics in photography, by Roger Ewing
Congruences, by Darold Comstock
Mechanical brains, by Richard Little.

The paper by Richard Little, *Mechanical brains*, was read by him at the combined meeting, in Ann Arbor, of the Student Section of the Michigan Section of the Mathematical Association of America and the regional chapters of *Kappa Mu Epsilon*.

Dr. Cleon C. Richtmeyer, of *Michigan Beta*, attended the *Kappa Mu Epsilon* Council meetings at Hofstra College during the Christmas recess.

Officers elected for 1950–51 are: President, Darold Comstock; Vice-President, Mary Bolla; Secretary, Doris Brode; Treasurer, Paul Buchholz; Corresponding Secretary and Sponsor, Prof. D. R. Sudborough. However, Prof. H. W. Zeoli will serve for Prof. Sudborough while the latter is on leave during the year.

Kappa Mu Epsilon, Baldwin-Wallace College

The following talks were presented to the *Ohio Gamma* chapter of *Kappa Mu Epsilon* during 1949–50:

Centrifugal force in industry, by Walter H. Craig

The mathematics of sound, by Dr. E. C. Unnewehr

The science and philosophy of mathematics, by Robert Miller and George Smolensky

Exponential functions and their applications to science, by Robert Noblit

The history of mathematics, by Theron Schwegler

Mathematics at NACA, by Miss Betty Hostetler

Quality control, by Dr. Fred Leone.

The annual banquet of *Kappa Mu Epsilon* was held in May at which Dr. Leone gave his paper.

The newly elected officers to serve for 1950–51 are: President, Bruce Glass; Vice-President, Richard Sutton; Secretary, Ruth Oberer; Treasurer, Barbara Ayres; Corresponding Secretary and Sponsor, Dr. Paul Annear.

Kappa Mu Epsilon, Wayne University

During the past year the *Michigan Gamma* chapter of *Kappa Mu Epsilon* took part in many activities to further interest in mathematics. The group has sponsored seven very interesting and informative talks on various aspects of mathematics. The topics and speakers were:

Newton's method of solving equations, by Dr. K. W. Folley

Foundations of mathematics, by Dr. R. Ackoff

The differential analyzer, by Dr. A. W. Jacobson

The calculus of variations, by Dr. Max Coral

Methods of solving equations, by Dr. T. Southard

Topology, by Dr. S. Kaplan

Gibb's phenomena, by Marvin Snyder.

The chapter attended a meeting at the University of Michigan in conjunction with the Michigan Section of the Mathematical Association of America. Faculty papers were presented at the morning session and six student members of *Michigan Gamma* presented papers during the afternoon session. Eight different institutions from Michigan and Ohio were represented with a total student attendance of 79. The conference proved so successful that the *Michigan Beta* chapter of *Kappa Mu Epsilon* at Mt. Pleasant, is sponsoring a similar meeting in the coming spring.

Twenty-seven new members were taken into the group at the annual initiation and banquet, at which Dr. G. Y. Rainich of the University of Michigan gave a talk on *Axioms*.

As a future project, the *Michigan Gamma* chapter hopes to hold joint meetings with the University of Detroit and the University of Michigan.

NEWS AND NOTICES

EDITED BY EDITH R. SCHNECKENBURGER, University of Buffalo

Readers are invited to contribute to the general interest of this department by sending news items to Edith R. Schneckenburger, University of Buffalo, Buffalo 14, New York. Items must be submitted at least two months before publication can take place.

INSTITUTE FOR TEACHERS OF MATHEMATICS

The Association of Teachers of Mathematics in New England is sponsoring an Institute for Teachers of Mathematics which will be held August 23–30, 1951 at Connecticut College, New London, Connecticut. Details of the program will be published in *The Mathematics Teacher*. If you wish a program, please drop a card to C. H. Mergendahl, Newton High School, Newtonville, Massachusetts or Katharine E. O'Brien, Deering High School, Portland, Maine.

NEW GRADUATE PROGRAM IN STATISTICS AT THE UNIVERSITY OF ILLINOIS

The University of Illinois has instituted a graduate program leading to the degree of Master of Arts or to the degree of Doctor of Philosophy in Statistics. Administered by the Department of Mathematics, the new program includes both statistical theory and its application to a specific field of interest. Considerable flexibility is possible in the division of emphasis between theory and its application but in no case will the amount of work in the latter area be less than that for a minor.

Under the new program a student who wishes to concentrate on research in statistical methods will have a choice of areas in which his thesis may be written, depending on his major interest. Each student will have two advisers, one in statistical theory and the other in his area of application, one of whom will be in charge of his thesis.

Beyond the regular Graduate College stipulations, the only special requirement for admission to the program is completion of a thorough course in calculus. For the Ph.D. degree a student, in addition to language and other standard requirements, must show capacity for important original contributions in statistical methods or in application of such methods.

ACKNOWLEDGMENT

The editor of the MONTHLY wishes to make grateful acknowledgment of the services rendered by the following persons who have refereed papers: R. P. Agnew, I. A. Barnett, P. M. Batchelder, Richard Bellman, Garrett Birkhoff, L. M. Blumenthal, C. B. Boyer, R. C. Buck, W. E. Byrne, G. F. Carrier, R. V. Churchill, N. A. Court, Wayne Dancer, Paul Erdös, Tomlinson Fort, V. G. Grove, William Gustin, E. H. C. Hildebrandt, Ralph Hull, B. W. Jones, P. S. Jones, Irving Kaplansky, E. P. Lane, Caroline Lester, V. O. McBrien, N. H. McCoy, A. D. Michal, Oystein Ore, George Pólya, G. B. Price, J. F. Randolph, John Riordan, Arthur Rosenthal, J. B. Rosser, H. N. Shapiro, I. M. Sheffer,

C. E. Springer, A. H. Taub, R. S. Underwood, H. E. Vaughan, P. M. Whitman, R. L. Wilder, H. E. Wolfe.

PERSONAL ITEMS

Professor H. F. Fehr of Teachers College, Columbia University, was the delegate of the Association at the Fifteenth Educational Conference, New York, on October 26–27, 1950.

Dean E. L. Mackie of the University of North Carolina represented the Association at the inauguration of President H. W. Tribble of Wake Forest College on November 28, 1950.

Professor N. H. McCoy of Smith College was the representative of the Association at the inauguration of President Spencer Miller, Jr. of American International College on November 8–9, 1950.

Professor R. F. Schnepp, St. Mary's University of San Antonio, was appointed to represent the Association at the dedication of Science Hall, Incarnate Word College, on November 3, 1950.

Professor C. R. Wylie, Jr. of the University of Utah represented the Association at the inauguration of President L. L. Madsen of Utah State Agricultural College on November 3, 1950.

Dr. R. V. Kadison of the University of Chicago has received an appointment to a National Research Fellowship in Mathematics for the year 1950–51.

Mr. Z. A. Melzak of McGill University was awarded the William Lowell Putnam Scholarship at Harvard University for the Tenth Annual Competition.

Augustana College reports: Associate Professor H. E. Nelson has been promoted to the position of Professor and Head of the Department of Mathematics; Professor W. E. Cederberg has retired with the title of Professor Emeritus.

Hope College announces: Assistant Professor J. E. Folkert has been granted a year's leave of absence for the purpose of doing graduate work at Michigan State College; Assistant Professor Harry Frissel of the Department of Physics has been made a temporary member of the department.

At Mississippi State College: Professor C. R. Pettis, head of the Department of Mathematics, and Associate Professor C. R. Stark have retired; Professor Arthur Ollivier has been named Head of the Department of Mathematics; Dr. L. H. Kanter of the University of Wisconsin has been appointed to an associate professorship.

At Washington Square College, New York University: Professor Wilhelm Magnas of California Institute of Technology has been appointed Research Associate Professor; Associate Professor C. K. Payne has retired.

Northern Illinois State Teachers College announces the following: Mrs. C. E. Hardgrove, formerly an instructor in the Department of Education, Ohio State University, and Dr. W. L. Pickard of the Naval Research Laboratory, Washington, D. C. have been appointed to associate professorships; Assistant Professor R. E. Anderson has been promoted to an associate professorship and has been granted a leave of absence while he is in military service.

Southern Illinois University announces the following appointments: Dr. A. M. Mark of the University of Wisconsin to the position of Assistant Professor of Mathematics and Director of University Statistical Service; Mr. Daniel Orloff, previously graduate student at the University of Chicago, to an instructorship.

Stevens Institute of Technology reports: Professor C. O. Gunther, formerly chairman of the Department of Mathematics, has retired with the title of Professor Emeritus; Assistant Professor M. R. Reeks, who is Chairman of the Department of Mathematics, has been promoted to an associate professorship; Mr. O. J. Karst, formerly of the Department of Physics, has been appointed to an associate professorship.

United States Naval Academy makes the following announcements: Associate Professor E. Hawkins has been promoted to a professorship; Assistant Professor J. F. Paydon has been promoted to an associate professorship; Professor G. R. Clements has retired; Professor J. B. Scarborough has retired with the title of Professor Emeritus; Assistant Professor K. F. McLaughlin has returned after spending two years' leave of absence at the University of Iowa.

At the University of Houston: Professor J. D. Hutchinson has been named Chairman of the Department of Mathematics; Assistant Professor Martin Wright has been promoted to an associate professorship.

The University of Maine reports the promotion of Associate Professor W. S. Lucas to a professorship and the appointment of Mr. G. F. Simmons, formerly teaching assistant at the University of Chicago, to an instructorship.

The University of Oklahoma announces: Dr. Y. W. Chen, formerly at the Institute for Advanced Study, has been appointed to an associate professorship; Assistant Professor A. E. Labarre, Jr., of the University of Idaho has been appointed to an instructorship; Associate Professor B. S. Whitney has been given the title of Director of the Observatory.

The University of Southern California announces the following: Dr. Bernard Sherman of the University of Vermont has been appointed to an assistant professorship; Associate Professor R. S. Phillips is on leave of absence and is at the Institute for Advanced Study.

University of Tennessee reports: Dr. Walter Baum of Eidgenossische Technische Hochschule, Zurich, and Dr. E. G. Kundert of Kantonsschule, Zurich, have been appointed to acting assistant professorships; the contract of the University with the Office of Naval Research has been extended for another year with Professor J. W. Givens and Mr. N. Heerema as investigators.

University of Vermont makes the following announcements: Assistant Professor J. A. Larrivee has been promoted to an associate professorship; Mr. William Baranoff, Mr. J. F. Detlef, Mr. W. T. Fishback, and Mr. J. E. Vollmer have been appointed to instructorships.

At the University of Wisconsin: Graduate Assistant L. E. Fuller has been promoted to an instructorship; Dr. J. B. Kelly is now at the Institute for Advanced Study.

The University of Wyoming announces the following: Assistant Professor V. J. Varineau has been promoted to an associate professorship; Mr. W. G. Brady has been granted a year's leave of absence to study at the University of Pittsburgh.

Wilson Junior College reports: Mr. J. H. Manheim has been appointed to an instructorship replacing Mr. Norman Stein who is now located at Fort Sheridan.

Mr. L. J. Abbeduto, a student at Illinois Institute of Technology, has been recalled to active duty in the Air Force and is located at Selfridge, Michigan.

Mr. D. L. Arenson is now a Research Engineer at Armour Research Foundation, Chicago, Illinois.

Dr. A. V. Baez of the Cornell Aeronautical Laboratory has been appointed Professor of Physics at University of Redlands.

Professor F. R. Bamforth of Ohio State University has been appointed to a professorship at Otterbein College.

Mr. M. Q. Barton, formerly a student at Central College, has received an appointment as graduate assistant at the University of Illinois.

Mr. H. G. H. Bartram, previously a graduate student at Cornell University, has been appointed to an assistant professorship at Coe College.

Associate Professor A. W. Boldyreff of the University of New Mexico has been appointed a member of the staff of Sandia Corporation, Albuquerque, New Mexico.

Mr. S. G. Bourne of Johns Hopkins University is now an instructor at the University of Connecticut and a member of the Institute for Advanced Study.

Dr. F. W. Brown of North American Aviation, Inc. has accepted a position as associate director for research and development, United States Naval Ordnance Test Station, Inyokern, California.

Mr. W. C. Brown, who was a graduate assistant at the University of Oklahoma, is now an actuarial student at Equitable Life Assurance Society, New York.

Dr. K. A. Bush has been appointed to an associate professorship at Champlain College.

Mr. J. A. Carpenter has accepted a position as mathematician with Snow and Schule, Inc., Cambridge, Massachusetts.

Mr. C. R. Carr, previously a teaching fellow at the University of Michigan, has received an appointment at Indiana Technical College.

Professor H. B. Curtis of Lake Forest College has retired with the title of Professor Emeritus.

Mr. D. L. Dunning, graduate assistant at Kent State University, is now Assistant Actuary at Warner-Watson, Inc., Chicago, Illinois.

Lecturer Albert Edrei of the University of Saskatchewan has been promoted to an associate professorship.

Mr. J. V. Finch of the University of Wisconsin has been appointed to an assistant professorship at Beloit College.

Assistant Professor Pearl L. Ford of Western Michigan College of Education has been promoted to an associate professorship.

Mr. R. F. Gabriel of St. Francis College has been appointed to an instructorship at University College, Rutgers University.

Mr. B. T. Goldbeck, Jr., of the University of Wyoming has received an appointment as instructor at Texas Christian University.

Dr. Helmut Grunsky of the University of Tubingen was Visiting Professor at State College of Washington for the first half of the academic year 1950–51.

Professor Marshall Hall, Jr. of Ohio State University has been recalled to active duty in the United States Navy.

Dr. H. J. Hamilton of Los Angeles City College has been appointed to an associate professorship at Pomona College.

Assistant Professor P. W. Healy of Southwestern College, Winfield, Kansas, has been promoted to an associate professorship.

Dr. Sarah B. Hill has been appointed to an assistant professorship at Wheaton College, Norton, Massachusetts.

Professor Aughtum S. Howard of Kentucky Wesleyan College has been appointed Acting Head of the Division of Science and Mathematics for 1950–51.

Dr. J. A. Jenkins has been appointed to an assistant professorship at Johns Hopkins University.

Mr. M. L. Lawson of the University of Oklahoma has been appointed to an associate professorship at Arkansas State College.

Dr. L. H. N. Lee, University of Minnesota, has been appointed to an assistant professorship at the University of Notre Dame.

Dr. Eugene Lukacs, who has been at the United States Naval Ordnance Test Station, Inyokern, California, has joined the staff of the Statistical Engineering Laboratory of the National Bureau of Standards.

Dr. Nathaniel Macon of the University of North Carolina is at the University of Amsterdam during 1950–51.

Mr. Fred Marer has been appointed Chairman of the Mathematics Department of Los Angeles City College.

Associate Professor R. H. Moorman, Tennessee Polytechnic Institute, has been appointed Acting Head of the Department of Mathematics.

Mr. C. H. Murphy, Jr. of the University of Hawaii has accepted a position as mathematician in the Ballistic Research Laboratory, Aberdeen Proving Ground, Maryland.

Assistant Professor P. B. Norman of Wagner College has been promoted to an associate professorship.

Mr. J. M. Patterson of Wayne University has been appointed to an instructorship at New York University.

Dr. C. J. Pipes of the University of Oklahoma has been appointed to an assistant professorship at Southern Methodist University.

Professor D. W. Pugsley, head of the Department of Mathematics of Berea

College, was on sabbatical leave during the first semester of 1950–51 while studying at Columbia University.

Assistant Professor A. O. Qualley of Drake University has been appointed Superintendent, Farnhamville Independent School, Iowa.

Mr. C. H. Reid, Jr., who has been teaching at Sargent Consolidated School, Monte Vista, Colorado, has been appointed to an instructorship at Fort Lewis Agricultural and Mechanical College.

Assistant Professor G. L. Tiller of Utica College, Syracuse University, has been appointed to an associate professorship at Southwestern at Memphis.

Dr. S. E. Urner has been appointed Professor and Chairman of the Department of Mathematics of the Los Angeles State College of Applied Arts and Sciences.

Professor C. E. VanHorn of Fisk University has been appointed Professor and Head of the Department of Mathematics of Wartburg College.

Professor Oswald Veblen has retired from active service in the Institute for Advanced Study.

Dr. J. E. Wilkins, Jr., formerly a mathematician at American Optical Company, Buffalo, New York, has accepted a position with the Nuclear Development Associates, Inc., New York City.

Mr. Herbert Wolf, previously a graduate student at the University of North Carolina, has been appointed to an assistant professorship at Winthrop College.

Dr. C. S. Yih of the University of British Columbia has received an appointment as associate professor at Colorado Agricultural and Mechanical College.

Mr. J. A. Zilber of Columbia University has been appointed to an instructorship at Johns Hopkins University.

Professor W. A. Ballou of the Department of Physics of Morehead State College died on November 20, 1950.

Professor M. M. Culver of the University of Pittsburgh died on November 2, 1950.

Dean W. E. Duckering of the University of Alaska died on October 7, 1950.

Professor R. O. Hutchinson, head of the Department of Mathematics of Tennessee Polytechnic Institute, died on October 22, 1950.

Sister M. Bertrand of Marywood College died on November 18, 1950.

THE MATHEMATICAL ASSOCIATION OF AMERICA

Official Reports and Communications

NEW MEMBERS

Professor H. M. Gehman, Secretary-Treasurer, announces that the following ninety persons have been elected to membership by the Board of Governors on applications duly certified.

A. H. ALBERT, M.S.(Michigan) Instructor, Kansas State Teachers College, Emporia, Kans.

J. S. AREND, S.B.(M.I.T.) Part-time Instructor, University of Colorado, Boulder, Colo.

H. H. BERRY, B.S.(Kentucky) 859 S. Broadway, Lexington, Ky.

A. H. BLUE, Ph.D.(Iowa) Professor, Culver-Stockton College, Canton, Mo.

S. R. BODNER, B.C.E.(Poly. Inst. of Brooklyn) Instructor, Polytechnic Institute of Brooklyn, N. Y.

C. W. BOSTICK, B.S.(M.I.T.) Grad. Student, University of Illinois, Urbana, Ill.

F. E. BOTHWELL, Ph.D.(M.I.T.) Asso. Professor, Northwestern University, Evanston, Ill.

HOLMES BOYNTON, Ed.D.(Columbia) Professor, Northern Michigan College of Education, Marquette, Mich.

T. F. BRIDGLAND, JR., B.S.(Florida) Grad. Assistant, University of Florida, Gainesville, Fla.

W. J. BURGESS, Faculty Member, Morris College, Sumter, S. C.

R. A. BURROWS, Student, Albion College, Mich.

VIRGINIA CARLOCK, A.B.(LaGrange C.) Teaching Fellow, Alabama Polytechnic Institute, Auburn, Ala.

W. E. CARSON, B.S.(Poly. Inst. of Brooklyn) Grad. Student and Part-time Instructor, Polytechnic Institute of Brooklyn, N. Y.

L. E. CARVILLE, M.A.(Boston U.) Mathematician, Army Map Service, Washington, D. C.

ELSIE T. CHURCH, M.A.(Kentucky) Instructor, University of Kentucky, Lexington, Ky.

C. R. CLARK, A.M.(Michigan) Research Assistant, Sandia Corporation, Albuquerque, N. M.

D. R. CLUTTERHAM, M.S.(Arizona) Grad. Assistant, University of Illinois, Urbana, Ill.

W. J. CODY, JR., Student, Elmhurst College, Ill.

E. L. DOLNEY, M.S.(Notre Dame) Instructor, University of Alaska, College, Alaska

ANDERSON DUGGAR, JR., Student, University of Detroit, Mich.

META M. EWING, M.S.(Michigan S. C.) Teacher, Bay City Junior College, Mich.

J. E. FAULKNER, B.S.(Utah S. C.) Grad. Assistant, Kansas State College, Manhattan, Kans.

D. H. FIRL, B.A.(Gustavus Adolphus C.) Grad. Assistant, Kansas State College, Manhattan, Kans.

R. J. FLANAGAN, B.S.(New Mexico) Mathematical Analyst, Sandia Corporation, Albuquerque, N. M.

J. R. FOOTE, Ph.D.(M.I.T.) Asst. Professor, Iowa State College, Ames, Iowa.

R. S. FOUCH, M.S.(Chicago) Asst. Professor, Arizona State College, Tempe, Ariz.

E. A. FRANZ, M.S.(Iowa) Instructor, Culver-Stockton College, Canton, Mo.

W. T. GLACKEN, Student, Purdue University, Lafayette, Ind.

V. D. GOKHALE, Ph.D.(Chicago) Professor, Atlanta University, Ga.

M. A. GOLUB, M.S.(New Brunswick) Instructor, University of Missouri, Columbia, Mo.

R. E. GRAVES, Ph.D.(Minnesota) Asst. Professor, University of Minnesota, Minneapolis, Minn.

W. R. GREANEY, Kingsbridge P. O., New York, N. Y.

E. V. GREER, M.A.(Illinois) Professor, Bethany-Peniel College, Bethany, Okla.

J. P. GWIN, Student, University of Tulsa, Okla.

SEYMOUR HABER, B.A.(Yeshiva C.) Part-time Instructor, Syracuse University, N.Y.

D. J. HANNAN, B.A.(Loras C.) Teaching Fellow, University of Detroit, Mich.

H. K. HARTMAN, B.S.(Bucknell) Claim Set-

tlement Agent, Department of Public Assistance, Harrisburg, Pa.

D. M. HOUSER, B.S. (Ottawa U.) Grad. Assistant, Kansas State College, Manhattan, Kans.

ERNEST IKENBERRY, Ph.D. (Louisiana State U.) Asst. Professor, Alabama Polytechnic Institute, Auburn, Ala.

R. G. INGLE, A.B. (Bethany-Peniel C.) Meteorologist, Sandia Corporation, Albuquerque, N. M.

R. T. JOHN, M.S. (Notre Dame) Clergyman and Instructor, Crosier Seminary, Onamia, Minn.

C. W. JORDAN, JR., B.A. (Williams C.) Asst. Professor, Williams College, Williamstown, Mass.

ABRAHAM KARRASS, Student, Brooklyn College, N. Y.

W. M. KOGER, Student, Southern Methodist University, Dallas, Tex.

E. R. LANCASTER, M.A. (Wayne) Instructor, University of Detroit, Mich.

G. N. LANDES, B.A. (Wooster) Mathematical Analyst, Sandia Corporation, Albuquerque, N. M.

E. V. LEWIS, Ph.D. (M.I.T.) Asst. Professor, University of Delaware, Newark, Del.

J. A. LEWIS, Ph.D. (Brown) Research Physicist, Corning Glass Works, N. Y.

G. O. LOSEY, Student, University of Michigan, Ann Arbor, Mich.

R. W. MALLEY, B.S. (Southeastern Louisiana C.) Mathematical Analyst, Sandia Corporation, Albuquerque, N. M.

J. H. MANHEIM, M.S. (Illinois) Instructor, Wilson Junior College, Chicago, Ill.

J. A. MANSOUR, B.S. (Detroit) Teaching Fellow, University of Detroit, Mich.

W. J. MCCLINTOCK, M.A. (Michigan) Teaching Fellow, University of Detroit, Mich.

GARNER MCCROSSEN, M.A. (Wyoming) Part-time Instructor, University of Colorado, Boulder, Colo.

J. E. MCLAUGHLIN, Ph.D. (C.I.T.) Instructor, University of Michigan, Ann Arbor, Mich.

S. A. MCLEOD, M.A. (North Carolina) Dean, Lander College, Greenwood, S. C.

G. W. MEDLIN, M.A. (North Carolina) Part-time Instructor, University of North Carolina, Chapel Hill, N. C.

E. P. MERKES, M.S. (DePaul) Lecturer, DePaul University, Chicago, Ill.

J. J. MILLER, B.S. in M.E. (New Mexico) Supervisor, Applied Mathematics Section, Sandia Corporation, Albuquerque, N. M.

MILTON MORRISON, M.A. (Columbia) Instructor, Stevens Institute of Technology, Hoboken, N. J.

J. H. MOSS, JR., M.S. (N.Y.U.) Asst. Professor, Alabama Polytechnic Institute, Auburn, Ala.

T. P. PALMER, A.M. (Harvard) Asso. Professor, Rose Polytechnic Institute, Terre Haute, Ind.

P. I. PETERS, M.A. (Ohio State) Asso. Professor, Union College, Barbourville, Ky.

R. B. PINSON, M.A. (Texas) Asso. Professor, Stephen F. Austin State College, Nacogdoches, Tex.

C. G. PITNER, Professor, Harding College, Searcy, Ark.

ETHEL H. RAYBOULD, M.A. (Queensland) Senior Lecturer, University of Queensland, Brisbane, Australia.

J. W. REED, B.S. (New Mexico) Supervisor, Meteorological Section, Sandia Corporation, Albuquerque, N. M.

D. F. RIDDLE, A.B. (Illinois) Grad. Student, University of Michigan, Ann Arbor, Mich.

R. A. SARTOR, Student, University of Detroit Mich.

JOSEPHINE SAVARO, Librarian, University of Scranton, Pa.

W. R. SEUGLING, M.A. (U.C.L.A.) Research Assistant, University of California at Los Angeles, Calif.

W. T. SHARP, M.A. (Toronto) Asst. Professor, University of Maryland, College Park, Md.

SISTER IRENE, M.A. (Catholic U.) Teacher, Annhurst College, Putnam, Conn.

SISTER M. CONSTANTIA, M.A. (Notre Dame) Head of Mathematics Department, Central Catholic High School, Toledo, Ohio.

D. O. SNOW, M.A. (Acadia) Asso. Professor, Acadia University, Wolfville, N. S.

SANDERS STONE, Student, Wilmington College, Undergraduate Center, Dayton, Ohio.

ANN A. TAYLOR, B.S. (Georgia) Teaching Fellow, Alabama Polytechnic Institute, Auburn, Ala.

P. M. TREUENFELS, M.S. (N.Y.U.) Mathematician, Ballistics Research Laboratory, Aberdeen Proving Ground, Md.

L. M. TULLOCH, A.M. (Brown) Asso. Professor, Southwest Texas State College, San Marcos, Tex.

G. E. UHRICH, Ph.D. (Washington) Asst. Professor, Montana State College, Bozeman, Mont.

C. M. VICK, Supervisor, Mathematical Analyst Section, Sandia Corporation, Albuquerque, N. M.

D. T. WALKER, B.S. (Wofford C.) Grad. Student, University of Georgia, Athens, Ga.

W. H. WARNER, B.S. (Carnegie) Teaching Assistant, Carnegie Institute of Technology, Pittsburgh, Pa.

S. E. WARSCHAWSKI, Ph.D. (Basel, Switzerland) Professor, University of Minnesota, Minneapolis, Minn.

W. J. WAYNE, Student, Butler University, Indianapolis, Ind.

G. P. WEEG, M.S. (Oklahoma A & M) Instructor, St. Ambrose College, Davenport, Iowa.

SONIA S. WOHL, M.S. (Columbia) Asst. Librarian, Rockefeller Institute for Medical Research, New York, N. Y.

F. A. YOKE, Student, San Diego State College, Calif.

W. M. ZARING, A.B. (Kentucky Wesleyan) Grad. Student, University of Kentucky, Lexington, Ky.

A. D. ZIEBUR, Ph.D. (Wisconsin) Asst. Professor, Oklahoma Agricultural and Mechanical College, Stillwater, Okla.

FEBRUARY MEETING OF THE LOUISIANA-MISSISSIPPI SECTION

The February meeting of the Louisiana-Mississippi Section of the Mathematical Association of America was held at Centenary College, Shreveport, Louisiana, February 24–25, 1950. Professor G. R. Trott, Chairman of the Section, presided.

Ninety-three persons attended the meeting, including the following thirty-nine members of the Association: T. A. Bickerstaff, L. Virginia Carlton, Margaret Davis, H. T. Donohoe, W. L. Duren, Virginia I. Felder, L. R. Ford, L. M. Garrison, M. E. Gillis, W. C. Griffith, B. H. Gundlach, J. A. Hardin, S. L. Hull, H. T. Karnes, C. G. Killen, Z. L. Loflin, A. C. Maddox, J. W. McClimans, Betty McKnight, R. A. Miller, B. E. Mitchell, S. B. Murray, Arthur Ollivier, R. L. O'Quinn, C. L. Perry, P. K. Rees, D. P. Richardson, F. A. Rickey, H. F. Schroeder, V. B. Temple, W. B. Temple, J. F. Thompson, B. B. Townsend, G. R. Trott, M. P. Tullier, Jr., A. D. Wallace, Eleanor B. Walters, M. C. Wicht, T. F. Wilbanks.

The following officers were elected for the coming year: Chairman, H. F. Schroeder, Louisiana Polytechnic Institute; Louisiana Vice-Chairman, T. F. Wilbanks, Southwestern Louisiana Institute; Mississippi Vice-Chairman, S. B. Murray, Mississippi State College; Secretary-Treasurer, F. A. Rickey, Louisiana State University.

By the invitation of the Executive Committee, Professor L. R. Ford delivered two addresses, one at the Friday evening dinner on the subject *The Superior Student*, the other at the Saturday morning session on *Some Remarkable Theorems about Areas*.

The following short papers were presented:

1. *Haversine-saversine trigonometry*, by B. E. Mitchell, Millsaps College.

The trigonometry of the haversine-saversine combination is analogous to that of the sine-cosine pair. The saver sine of an angle is defined as the haversine of the supplement of the angle. The angle being θ, the symbols employed are hav θ and sav θ. The fundamental relation (identity) connecting them, hav θ + sav $\theta = 1$, is indicative of the simplicity of their relations. Their definitions

as lines, their graphs and goniometry, calculus, development into series, and their use in solving both plane and spherical triangles are discussed.

2. *Extensional invariance*, by A. D. Wallace, Tulane University.

Any completely regular Hausdorff space has a homeomorph X densely contained in a compact Hausdorff space X' so that (denoting closure in X' by $'$) if A, B are closed in X then $(A\cap B)' = A'\cap B'$ provided that either (i) X is normal, or (ii) X is locally compact and one of A, B is compact. This is merely the Tychonoff imbedding (as improved by Cech and Stone) together with the additional result about closure. Case (i) can be deduced from some results of Wallman but case (ii) cannot, since Wallman's compaction is not Hausdorff unless X is normal. This result is applied to prove the extensional invariance of certain properties of X. For example unicoherence, if X has property S.

3. *Some variations in the content of an analytic geometry course*, by L. Virginia Carlton, Northwestern State College.

In an effort to make the concepts of plane analytic geometry more fruitful, the author of this paper proposes two variations—the teaching of polar coordinates simultaneously with rectangular coordinates, and greater use of determinants. Based on the recent study of trigonometry, the polar approach is not only clearer in itself, but by contrast clarifies the advantages of the rectangular system. Three of the many uses of determinants are discussed, namely the derivation of the formula for the distance from a point to a line, the determination of the turning point of a quadratic function, and the characteristic determinant in eliminating the xy-term from the general second-degree equation.

4. *On metric extensors*, by B. B. Townsend, Louisiana State University.

"Metric" extensors, $g_{\alpha\alpha\beta\beta}$ and $g^{\alpha\alpha\beta\beta}$, are obtained from the Riemannian metric tensor g_{ab} and its contravariant analogue g^{ab}. These extensors enjoy many of the analogous properties in extensor theory as do the corresponding tensors in tensor theory. In particular, full range contractions on upper and lower doublet indices yield the product of two Kronecker deltas. This property makes possible the use of the well known technique of raising and lowering indices.

5. *On simplification of the equation of a quadric surface*, by S. L. Hull, University of Arkansas.

This paper deals with a method of removing the cross-product terms from the equation of an ellipsoid, without using rotation formulas. The equation is written in the form of a symmetric matrix. A parameter λ is subtracted from each element of the main diagonal of its determinant, and the resulting cubic equation is solved. The validity of this introduction of λ is established by a consideration of the gradient on an axis of the ellipsoid. The three roots of this cubic equation are the coefficients in the new equation having no cross product terms. A numerical example is solved to illustrate the method.

6. *Laplace transform and boundary value problems*, by Margaret LaSalle, Louisiana State University, introduced by F. A. Rickey.

The object of the theory as developed by H. S. Carslaw and J. C. Jaeger is to show the use of the Laplace transform in the solution of the partial differential equation $v_2 = v_{11}$, where $v = f(x, t)$. By use of the Laplace transform, this is reduced to a total differential equation, which is solved by usual methods. The Fourier-Mellin inversion theorem and contour integration are used in determining the particular solution satisfying the initial conditions of the problem.

7. *Two versions of spectral theory*, by W. L. Duren, Tulane University.

The proof of the equivalence of two definitions of the spectrum of a linear transformation is presented in this paper. The classical definition asserts that λ is in the spectrum of the transforma-

tion T in an algebra of linear transformations if and only if $T-\lambda I$ has no inverse in $\mathfrak{A}$. A modern form of spectral theory arises from an isomorphism of $\mathfrak{A}$ into the algebra of all continuous complex valued functions on some compact Hausdorff space. The image of T is a function whose range is called the spectrum of T in the second definition.

8. *The testing program*, by H. T. Karnes, Louisiana State University.

This was the second in a series of annual reports of the testing program as given by the departments of mathematics in the colleges of Louisiana and Mississippi to entering Freshmen. The report was for the year 1949–50. There was an indication that the 1949–50 class was somewhat better than the 1948–49 class. This could have been due to uncontrolled factors. In view of this, it was decided to administer the test for at least one more year.

F. A. RICKEY, *Secretary*

APRIL MEETING OF THE IOWA SECTION

The Iowa Section of the Mathematical Association of America held its 37th annual meeting at the State University of Iowa, Iowa City, Iowa, on Friday and Saturday, April 21, 22, 1950. The Chairman, Professor B. E. Gillam of Drake University, presided.

One hundred and four persons attended the meeting, including the following forty-eight members of the Association: E. W. Anderson, W. S. Bicknell, E. W. Chittenden, B. B. Clark, Byron Cosby, Jr., A. T. Craig, Marian E. Daniells, W. M. Davis, R. M. Deming, Rev. L. E. Ernsdorff, A. M. Feyerherm, L. R. Ford, R. W. Gardner, R. E. Gaskell, B. E. Gillam, Cornelius Gouwens, Clara L. Hancock, F. S. Harper, Gertrude A. Herr, J. J. L. Hinrichsen, Helen P. Hoffman, D. L. Holl, G. E. Kaldenberg, Dora E. Kearney, L. A. Knowler, O. C. Kreider, R. J. Lambert, C. E. Langenhop, Ta Chung-Heng Li, Mary Beth Lieberknecht, F. W. Lott, Jr., R. B. McClenon, J. V. McKelvey, Martha M. McKelvey, C. G. Maple, W. H. Marlow, E. E. Moots, E. N. Oberg, H. V. Price, R. W. Raymond, Fred Robertson, Hazel M. Rothlisberger, M. F. Smiley, F. M. Stein, O. R. Taylor, H. P. Thielman, Henry Van Engen, Roscoe Woods,

Professor Van Engen reported for the committee to formulate by-laws for the section. The recommendations of the committee were adopted by an unanimous vote.

The following members were elected to hold office for the ensuing year: Chairman, Professor D. L. Holl, Iowa State College; Vice-Chairman, Rev. L. E. Ernsdorff, Loras College; and Secretary-Treasurer, Professor Fred Robertson, Iowa State College.

The following papers were presented at the meeting:

1. *Early history of the Iowa section*, by Professor R. B. McClenon, Grinnell College.

Professor McClenon gave a short account of the organization of the Iowa Section and of its first ten regular meetings. The Section held regular fall and spring meetings for a good many years, but in recent years it has met annually with the Iowa Academy of Science. The Section has shown a very gratifying growth in numbers and in strength with the passing years.

2. *Bounds for the derivatives of the solution of the Neumann problem*, by Professor C. G. Maple, Iowa State College.

The hypercircle method in function space introduced by Pfager and Synge is used to obtain bounds at a point for the derivatives of the solution of the Neumann problem. It is assumed that the hypercircle in function space on which the solution vector is located has already been determined. Then a free Green's vector is defined, and bounds fixed for the scalar product of the solution vector and the Green's vector. This same scalar product is also expressed in terms of certain calculable integrals and the value of a derivative of the solution function at an interior point. A combination of these two results gives bounds for the derivative of the solution at any point interior to the domain of definition of the problem. Only first and second order derivatives are considered.

3. *Note on Levinson's existence theorem for forced periodic solutions of a second order differential equation*, by C. E. Langenhop, Iowa State College.

Levinson (N. Levinson, *Journal of Math. and Phys.*, vol. 22, 1943, p. 41) has proved the existence of a periodic solution of the equation

$$\ddot{x} + f(x, \dot{x})\dot{x} + g(x) = e(t)$$

when $e(t)$ is periodic, and $f(x, \dot{x})$ and $g(x)$ satisfy certain rather general conditions, among which is $\lim_{x\to\infty}[g(x)/G(x)]=0$, where $G(x)=\int_0^x g(x)dx$. By altering Levinson's proof slightly at a critical point, the existence can be shown when this condition is replaced by $\lim_{x\to\infty} \sup [g(x)/x]$. These two conditions cover all possible $g(x)$ satisfying the other requirement on $g(x)$, i.e. $\lim_{x\to\pm\infty} g(x) = \pm\infty$. It does not appear to be possible to treat both cases the same way throughout.

4. *Potential flow into circumferential openings in drain tubes*, by Don Kirkham, Iowa State College, introduced by the Secretary.

A theoretical analysis of the effect of the spaces between drain tube units as used in the artificial drainage of soil is given. The problem is one of potential flow; therefore, the results are applicable to heat flow, etc. The basic problem solved is that for axially symmetric flow from an external cylindrical boundary at constant potential to a series of equal, equally-spaced openings at a lower potential, all located axially on, and comprising a part of, the otherwise impervious drain tube. The radii of the open sections and impermeable sections of the drain tube are equal. The basic problem is extended to obtain the solution to the practical problem of the seepage of groundwater into drain tubes beneath a horizontal watertable. The exact solution of the basic problem is obtained, but it is not suitable for numerical work. Accordingly, approximate solutions of specified uncertainty are derived and are utilized for tabulation of numerical results. As an example, the analysis shows, in the case of 6 inch diameter drain tubes, having 1 foot long impermeable sections and buried 4 feet deep in uniformly permeable soil, that increasing the openings from 1/32 inch width to 1/4 inch width will increase the flow 36 per cent; while embedding the tubes in gravel, to make the 1/32 inch openings of effectively infinite width, will increase the flow 180 per cent.

5. *Elastic deflection of a split ring*, by E. W. Anderson, Iowa State College.

The investigation is concerned with the stresses and deflections developed in split rings due to the action of circumferentially applied radial loads which are distributed symmetrically with respect to a diametral axis through the gap. Assuming that a condition of plane stress exists, biharmonic Airy's stress functions in Fourier series form are used to satisfy equilibrium and boundary conditions.

6. *On functional equations, generalized logarithms, and commutative functions*, by H. P. Thielman, Iowa State College.

Necessary and sufficient conditions on the rational, integral function $p(x, y)$ were given in order that the functional equation $F[p(x, y)]=F(x)+F(y)$ might have continuous, monotone solu-

tions. The functional equations $f(xy)=y^{\alpha}f(x)+x^{\alpha}f(y)$, and $g(xy)=\alpha g(x)\cdot g(y)+G(x)+g(y)$ were used to define generalized logarithms. These functional equations and their solutions were used to construct one-parameter continuous groups of commutative functions. Two functions $f(x)$ and $g(x)$ were defined to be commutative with each other if $f[g(x)]=g[f(x)]$. A particular example of the one parameter groups considered was $f(\gamma, x)=(\gamma x^{\alpha}+1-\gamma)^{1/\alpha}$ where γ is the parameter. Here $[\gamma, f(\beta, x)]=f(\gamma\beta, x)$. The identity of the group was given as $f(1, x)=x$.

7. *Use of complex variable in plane rigid motion*, by Professor L. R. Ford, Illinois Institute of Technology.

This was an invited address. By the use of complex variables, Professor Ford discussed some of the motions and some of the properties of the configurations thus generated by a point in a smooth plane sliding over a smooth stationary plane.

8. *Bernoulli numbers*, by Professor R. B. McClenon, Grinnell College.

This paper by Professor McClenon was concerned with the origin of these numbers in the *Ars Conjectandi* of James Bernoulli. Later work, by Euler especially, was referred to, and the usefulness of the Bernoulli numbers in many expansions was shown in the case of trigonometric and other functions.

9. *Measures for predicting calculus achievement*, by O. C. Kreider, Iowa State College.

Seven hundred eighty-nine algebra students chose curriculums requiring calculus; only 552 finished first-quarter calculus at the Iowa State College. Biserial r was used to test measures of mortality for the students who did not persevere through calculus. Correlation coefficients, regression equations, F and t tests were used to test measures for predicting scholastic achievement in calculus.

Mortality can be forecast reasonably well from the first-quarter all-college quality point average. Prediction of a calculus mark can be done reasonably well by using the algebra achievement test score, algebra course mark, and all-college average. All these measures were taken at the end of the students' first quarter in college.

10. *Note concerning the coefficients of the moments of the Bernoulli distribution*, by Dorothy DeWitt, Iowa State Teachers College, introduced by the Secretary.

In this note it is proved that the coefficients of the Bernoulli or binomial distribution are related to the differences of zero. A simple method for determining the coefficients of the rth moment from those of the $(r-1)$st moment is given.

11. *Note on a theorem of Miquel*, by Roscoe Woods, State University of Iowa.

This note is concerned with the envelope of a side, say DE, of the Miquel triangle DEF associated with the Miquel point M. The locus of M is also considered when DE is revolved about a fixed point in the plane. Conditions that the locus M be a circumconic of the reference triangle are considered in detail.

12. *On the stochastic independence of a ratio and its denominator*, by R. V. Hogg, State University of Iowa, introduced by the Secretary.

The purpose of this paper is to obtain a necessary and sufficient condition that the random variables y and x/y be stochastically independent. If $M(t_1, t_2)\equiv E[\exp(t_1x-t_2y)]$ be the moment generating function of the two dimensional random variable (x, y), this condition is, under appropriate restriction, that

$$\frac{\partial^k M(0,0)}{\partial t_2^k} \cdot \frac{\partial^k M(0,t_2)}{\partial t_1^k} \equiv \frac{\partial^k M(0,0)}{\partial t_1^k} \cdot \frac{\partial^k M(0,t_2)}{\partial t_2^k}$$

for $k=0, 1, 2, \cdots$.

13. *Some properties of quotients of additive set functions*, by E. W. Chittenden, State University of Iowa.

This paper was read by title.

14. *On the transcendence of* e^x, by Ta Chung-Heng Li, Drake University.

Let $A(z)$, $B(z)$, $a(z)$, $\beta(z)$, and $H(z)$ denote integral functions with rational Taylor's coefficients. For given $A(z)$ and $B(z)$ there exist "uniquely" $a(z)$ and $\beta(z)$ such that

$$A(z)a(z) + B(x)\beta(x) \equiv H(x)$$

where $H(z)$ denotes the H. C. F. of the given functions. If $B(z)$ is "irreductible," and $A(z)$ has one zero in common with $B(z)$, then $A(z)$ is divisible by $B(z)$. By virtue of this fact, we can easily show that e^x is transcendental for every algebraic value of x.

15. *Constant rank matrices*, by H. D. Mills, Iowa State College, introduced by the Secretary.

The matrix polynomial

$$A(t) = A_0 + A_1 + \cdots + A_k t^k$$

where the elements of A_i are in a field F, and t is a scalar variable in a subfield F^1 of the algebraic closure of F, is said to be of constant rank r over F^1 if $A(t)$ is exactly of rank r for every t in F. If F^1 contains more than $k(r+1)$ distinct elements, a necessary and sufficient condition for $A(t)$ is to be of constant rank r over F^1 is that A_0 be of rank r and $D_1(t)$ the first invariant factor of $A(t)$ have no zero in F^1.

If $A_s=0$, $s=2, 3, \cdots$, and $A(t)$ is of constant rank r, $A(t)$ can be put in the canonical form,

$$PA(t)Q = \begin{pmatrix} I + B_1 t & B_2 t \\ B_3 t & 0 \end{pmatrix},$$

where P, Q and B_i are constant, I and B_1 are $r \times r$ matrices, and

$$B, B_1^m B_2 = 0 \qquad m = 0, 1, 2, \cdots$$

with certain other conditions, the last equation is shown to be necessary and sufficient for $A(t)$ to be of constant rank r.

16. *Panel discussion: The teaching of college algebra and trigonometry* by J. V. McKelvey, Iowa State College (Chairman), Henry Van Engen, Iowa State Teachers College, H. V. Price, University High School, Iowa City, Iowa, Viola Smith, High School, Maquoketa, Iowa, and L. W. Swanson, Coe College.

This discussion included such topics as the solution of word problems, intelligent reading of text material, accurate use of mathematical symbols, mathematics courses for liberal arts and engineering students, algebraic and trigonometric identities, the solution of simultaneous linear equations, and testing of entering freshmen. A brief question and answer period closed the discussion.

The Iowa Academy of Science offers a $50 prize for the best paper presented at its annual meeting. Each section submits the non-invited paper considered best on its program, and the prize winning paper is selected from this group.

As the Iowa Section meets jointly with the Iowa Academy of Science, the papers presented at this meeting are eligible to compete for the prize. The committee selected the paper by Professor E. W. Anderson on *Elastic deflection of a split ring* as the entry from Mathematics.

FRED ROBERTSON, *Secretary*

APRIL MEETING OF THE ROCKY MOUNTAIN SECTION

The thirty-third annual meeting of the Rocky Mountain Section of the Mathematical Association of America was held at the University of Denver, Denver, Colorado, April 28 and 29, 1950. There were three sessions with Professor A. J. Lewis presiding at each.

The meeting was attended by approximately one hundred persons including the following fifty-three members of the Association: R. V. Anderson, C. F. Barr, D. L. Barrick, W. G. Brady, W. E. Briggs, J. R. Britton, R. L. Calvert, R. C. Campbell, F. M. Carpenter, F. L. Celauro, Nancy V. H. Cheney, A. G. Clark, G. S. Cook, G. A. Culpepper, L. C. Dawson, David DeVol, J. R. Everett, O. J. Falkenstern, A. B. Farnell, F. N. Fisch, Katherine C. Garland, H. T. Guard, R. R. Gutzman, Leota C. Hayward, I. L. Hebel, LeRoy Holubar, Burrowes Hunt, C. A. Hutchinson, B. W. Jones, Claribel Kendall, A. J. Lewis, C. C. MacDuffee, J. C. McKenzie, W. K. Nelson, Greta Neubauer, K. L. Noble, D. O. Patterson, H. C. Peterson, A. W. Recht, L. W. Rutland, Jr., Nathan Schwid, S. R. Smith, W. N. Smith, L. C. Snively, M. E. Sperline, K. H. Stahl, J. M. Staley, P. O. Steen, J. F. Stockman, E. P. Tovani, V. J. Varineau, W. W. Varner, Lillie C. Walters.

At the business meeting, the following officers were elected for the coming year: Chairman, Professor D. O. Patterson, Colorado State College of Education; Vice-Chairman, Professor F. L. Celauro; Secretary-Treasurer, Professor J. R. Britton, University of Colorado.

The following program of papers was presented:

1. *A note on operators*, by Mr. H. C. Peterson, University of Denver.

2. *A nonlinear differential equation of heat conduction type*, by Professor Nathan Schwid, University of Wyoming.

The solution of the differential equation for the flow of heat in one direction, when the thermal conductivity K and the specific heat C each is of the form $\alpha+\beta u$, where u is the temperature and the ratio β/α is small, was considered for a semi-infinite and for a finite bar. With suitable boundary conditions a solution can be obtained if the ratio K/C depends upon the temperature.

3. *Some properties of Fibonacci sequences*, by Mr. David DeVol, University of Colorado.

Defining Fibonacci sequences by the property $u_{n+1}=u_n+u_{n-1}$, several relations between the terms are easily obtained by the manipulation of two-by-two matrices whose elements are terms of the sequence. The speaker concluded by pointing out a geometric connection between the Fibonacci sequences and the sequences of polygonal numbers.

4. *Determination of a class of solvable biquartic equations*, by Professor L. C. Dawson, Colorado A and M College.

Solvable biquartic equations of the form

$$x^8 + ax^6 + bx^5 + cx^4 + dx^3 + ex^2 + fx + g = 0$$

may be formed by assigning certain real values to the coefficients. We impose the condition that the general biquartic be expressed as a difference of two squares, thus reducing the given biquartic to two quartics each solvable by known methods. This procedure yields two necessary conditions: $a^2-4c=0$ and $(a+b)^2-4(c+d+e+f+g)=0$, whereby the coefficients may be chosen so that the biquartic decomposes into a pair of quartics. A similar procedure is applicable to the determination of a class of solvable bicubics.

5. *A problem in the theory of runs*, by Professor A. G. Clark, Colorado A and M College.

In a set of independent trials of an event where the probability of a specified outcome is constant, an asymptotic expression was obtained for P_n, the probability that a run of given length will result for the first time with the nth trial. With this definition of n, $E(n)=\sum_{i=1}^{n} P_i=1/2$. Furthermore, $E[n-E(n)]^2$ was determined as a measure of the lack of stability of n. Attention was focussed on the extent to which the solution by elementary methods of this problem in the classical theory of probability makes use of subject matter pertinent to nearly every course offered in the usual undergraduate curriculum in mathematics.

6. *A note on the calculation of residues*, by Professor C. A. Hutchinson, University of Colorado.

The expression for the residue of an analytic function at a pole of order n is obtained as a determinant of order $n-1$. In an illustrative example, the determinant is evaluated by means of a second-order linear difference equation.

7. *Linear equations without determinants*, by Professor C. C. MacDuffee, University of Wisconsin.

8. *Cross-purposes in education*, by Professor C. C. MacDuffee.

This was an evening address at which Professor MacDuffee was the guest speaker.

9. *Progress in mathematics by the U.S.S.R. since World War II*, by Mr. R. J. Howerton, Regis College.

Since January 1948, all Russian scientific journals have been published in the languages of Russia only. A survey was made of the titles and authors of the papers appearing in the six leading mathematical journals of the U.S.S.R. for 1948–49. Four of these were carried back through 1947 and one, *Akademiya Nauk, S.S.S.R., Doklady*, was carried back through 1946 since it carried the greatest number of papers for 1947–48. A classification was then made of the papers into six general categories of mathematics. The following conclusions were drawn: (1) There was a general increase of activity in 1949 over 1948, the greatest increase being shown by topology and group theory; (2) The most profitable journal for an American (Russian reading) would be *Akademiya Nauk, S.S.S.R., Doklady*, unless he were in the field of applied mathematics, in which case *Prikladnaia Matematika i Mechanika* would be the most fruitful; (3) Due to the difficulty caused by transliteration from the Latin alphabet to the Russian and back again, no conclusive evidence was obtained to show an increase in the number of Germanic names among the authors of papers; (4) The work of the Russians seems to be of the highest quality and would do credit to any American Journal. (The same results were obtained by Mr. Paul W. Howerton in the field of organic chemistry. See *Russian literature in the field of organic chemistry*, Journal of Chemical Education, April, 1949); (5) Several writers have turned out a large volume of work, the most prolific being N. G. Chebotarev, with ten papers in two years; (6) There is no evidence of any political slant to any of the papers read.

10. *Problems in the training of teachers of mathematics*, Professor A. W. Recht, University of Denver.

After the program of papers, a joint meeting was held with the Mathematics Section, Eastern Division, Colorado Education Association. The discussion was concerned with the formation of the Colorado Council of Teachers of Mathematics.

J. R. BRITTON, *Secretary*

MAY MEETING OF THE INDIANA SECTION

The twenty-seventh annual meeting of the Indiana Section of the Mathematical Association of America was held at Wabash College, Crawfordsville, Indiana, on Saturday, May 6, 1950. Two sessions were held at which Professor Ralph Hull of Purdue University, Chairman of the Section, presided.

There were sixty-two in attendance including the following thirty-six members of the Association: Juna L. Beal, L. G. Black, Stanley Bolks, C. F. Brumfiel, G. E. Carscallen, W. W. Chambers, T. E. Cheatham, H. E. Crull, M. W. DeJonge, V. E. Dietrich, P. D. Edwards, W. R. Fuller, E. L. Godfrey, Michael Golomb, S. H. Gould, G. H. Graves, J. R. Hadley, N. R. Hughes, Ralph Hull, M. W. Keller, E. L. Klinger, R. A. Lufburrow, R. B. Merrill, P. T. Mielke, P. M. Nastocoff, C. C. Oursler, P. W. Overman, Philip Peak, J. C. Polley, Arthur Rosenthal, M. E. Shanks, Jane A. Uhrhan, R. O. Virts, J. L. Wilson, Florence A. Wirsching, W. D. Wood.

The following officers were elected: Chairman, H. E. Crull, Butler University; Vice-Chairman, M. W. Keller, Purdue University; Secretary, J. C. Polley, Wabash College.

On the matter of awarding Association medals as prizes in high school mathematics contests the chairman was authorized to appoint a committee with power to act. The committee was instructed to investigate the possibility of making such awards in connection with the Indiana State Mathematics Contest and the Indiana Science Talent Search.

The annual meeting of 1951 will be held on Saturday, May 5, the place of meeting to be announced later.

The following papers were presented:

1. *Mathematics for engineers*, by Professor M. E. Shanks, Purdue University.

Of two significant trends in mathematics for freshmen, terminal courses designed solely to fill the cultural gap, and a unified non-compartmentalized course in algebra, trigonometry, and analytic geometry, in part cultural but chiefly motivated by a need for bringing so called advanced ideas down into the undergraduate program, the latter was emphasized. In the author's opinion the need of the modern engineer for the advanced ideas, for pure mathematics, is essential, and once the engineer recognizes that the less traditional course could clearly increase his mathematical "power" he would welcome the change.

2. *A proof of the existence of a real zero for a polynomial of odd degree with real coefficients which is not dependent on continuity*, by Professor J. C. Polley, Wabash College.

This proof of the existence of a real zero for the polynomial $P(x)=\sum_{i=0}^{n}a_ix^{n-i}$, where the a_i are real, and $a_0>0$, is based on the theorems proving the existence: (1) of a number A such that, for all $x>A$, $P(x)>0$; (2) of a number B such that, for all $x<B$, $P(x)<0$; (3) of a number $d>0$ such that, for all x for which $0<x\leqq d$ and $-d\leqq x<0$, $|p(x)|=\sum_{i=0}^{n-1}a_ix^{n-i}<D$, where D is any positive number, however small. The proof consists in showing that at $x=c$, the least x, such that $P(x)>0$ for all greater x, $P(x)$ can be neither positive nor negative, whence, being defined, it must vanish.

3. *The crystallographic groups*, by Mr. C. L. Hassell, Purdue University, introduced by Professor Ralph Hull.

The 32 classes of crystallographic groups were described in terms of their representation as subgroups of the full orthogonal groups of three dimensions. It was pointed out that eleven of them are subgroups of the proper orthogonal groups, others are obtained from those of the first kind by adjoining the central reflection, and the rest are obtained from those of the first kind in another way. Illustrations were given for many of the classes, and it was mentioned that apparently no crystal substance is known for one of the classes, or at least this was the case up to 1938.

4. *On the content of the first course in mathematical statistics*, by Professor C. F. Kossack, Purdue University, introduced by Professor Ralph Hull.

As a first course in mathematical statistics is commonly taught the mathematics is superimposed upon a standard course in statistical methods, each derivation being treated separately and somewhat isolated from the others, whence the student fails to appreciate the theoretical basis of statistical methods. A course should be developed stressing the logical basis of the methods and the inductive approach to problems, introducing the elements of probability as needed and approaching statistical theory from at least a semi-axiomatic basis. The main ideas of hypotheses, testing, etc., should be stressed, and the student's material restricted to the simpler illustrations which he can handle mathematically.

5. *Linear graphs and the economics of transportation*, by Professor Tjalling C. Koopmans, Director of Research of the Cowles Commission, University of Chicago.

This was an invited address. The theory of linear graphs was applied to finding the most economical routing plan for transportation equipment (ships, say) in carrying out a given program consisting of constant monthly cargo flows between each pair (i, j) of n ports. If, for $i, j=1, \cdots, n$, y_{ij} are cargo flows in shiploads per month, x_{ij} flows of empty ships, t_{ij} and s_{ij} average times for loading and empty movement respectively, b_i the monthly surplus of ships, and Z the amount of shipping required for a routing plan y_{ij}, x_{ij}, then $b_i=\sum_j y_{ji}-\sum_j y_{ij}$ and $b_i=\sum_j x_{ij}-\sum_j x_{ji}$ where $x_{ij}\geqq 0$ (thus defining a convex polyhedral set). Hence $Z=\sum_{ij}t_{ij}y_{ij}+\sum_{ij}s_{ij}x_{ij}=Y+X$, say, where X is a linear function of x_{ij} which has at most one minimum anywhere in the set, but possibly at more than one point.

With any point x_{ij} minimizing X exists a non-negative number set p_{ij} representing nominal prices for a unit of transportation services, and a set p_i representing evaluations of the location of a ship under efficient routing, such that $p_{ij}-p_i+p_j-t_{ij}\leqq 0$ and $-p_i+p_j-s_{ij}\leqq 0$ for all (i, j), the former being zero for $y_{ij}>0$, and the latter zero for $x_{ij}>0$. The p_{ij} and p_i are the same for all programs permitting an efficient routing plan for which the linear graph G, consisting of all routes (i, j) such that $x_{ij}>0$, is the same.

Since the p_{ij} are reflected in freight rates in a perfectly competitive market, the analysis applies in developing a system of rates for a regulated transportation system so as to induce social efficiency in the choice of industrial locations by private entrepreneurs.

J. C. POLLEY, *Secretary*

ANNUAL MEETING OF THE MINNESOTA SECTION

The annual meeting of the Minnesota Section of the Mathematical Association of America was held at Macalester College in St. Paul, Minnesota, on Saturday, May 6, 1950. A session was held in the forenoon followed by a joint luncheon and afternoon session with the Minnesota Council of Teachers of Mathematics. Professors E. J. Camp, W. L. Hart and Kenneth May (Chairman of the Section) presided at the respective sessions. Professor J. R. Mayor of the University of Wisconsin was the principal speaker at the luncheon.

One hundred and thirty persons attended the meeting, including the following sixty-eight members of the Association: H. M. Anderson, F. J. Arena, J. E. Bearman, E. J. Berger, K. H. Bracewell, L. E. Bush, W. H. Bussey, R. H. Cameron, E. J. Camp, H. D. Colson, W. F. Crum, K. E. Dubbert, Helen Engebretson, I. C. Fischer, Walter Fleming, G. C. Francis, C. B. Germain, H. W. Godderz, Ruby M. Grimes, K. L. Hankerson, Charles Hatfield, J. Stanley Hill, J. E. Hafstrom, Hildegarde Horeni, G. H. Jaeger, D. A. Johnson, Reverend W. C. Kalinowski, G. K. Kalisch, Karlis Kaufmanis, P. G. Kirmser, L. S. Laws, R. L. Lokensgaard, W. S. Loud, Walter Lyche, H. B. MacDougal, W. R. McEwen, R. B. McHugh, C. R. McIntosh, K. W. McVoy, S. L. Mason, Kenneth May, J. R. Mayor, Sigurd Mundljeld, M. J. Norris, E. P. Northrop, F. R. Ohnsorg, Margaret Owchar, G. C. Priester, Mrs. Ruth Scholten, Dorothy V. Schrader, S. C. Simonson, Sister Ada Marie, Sister M. Joanne, Sister Mary Leontius, Sister Mary Seraphim, F. C. Smith, R. C. Staley, Irwin Stoner, A. G. Swanson, F. J. Taylor, Takashi Terami, Matilda B. Thompson, Marian W. Thornton, H. L. Turrittin, E. C. Varnum, O. E. Walder, K. W. Wegner, Irene L. Wente.

The nominating committee, consisting of Professor M. J. Norris, Chairman, Professor H. L. Turrittin, and Professor Kenneth Wegner, presented the following slate of nominees for the coming year: Chairman, J. M. H. Olmsted, University of Minnesota; Secretary, L. E. Bush, College of St. Thomas; Executive Committee, Kenneth May, Carleton College; Walter Fleming, Mankato State Teachers College; H. M. Anderson, Gustavus Adolphus College. These were duly elected. The following resolutions were passed: (1) That the term of office of all appointed committees shall end at the annual meeting; (2) That the committees on the high school mathematics contest and on cooperation with the Minnesota Council of Teachers of Mathematics be continued for another year.

By invitation of the Executive Committee, Dean E. P. Northrop of the University of Chicago delivered an address at the morning session on *The Role of Mathematics in General Education.*

The following seven short papers were presented:

1. *On two circular-hyperbolic functions and their application to differential equations*, Professor F. J. Arena, North Dakota State College.

Professor Arena discussed the properties of the functions

$$c(x) = \sum_0^\infty \frac{x^{2b}}{(2b)!}, \qquad s(x) = \sum_0^\infty \frac{x^{2b}}{(2b+1)!},$$

and showed how the use of these functions simplifies the solution of the linear differential equation of the second order with constant coefficients.

2. *An extension of Schwarz's lemma*, Professor Irwin Stoner, University of Minnesota.

In order to prove Riemann's mapping theorem, H. A. Schwarz formulated the lemma which states that if $f(z)$ is analytic, regular for $|z|<1$, if $|f(z)|<1$ for $|z|<1$, and if $f(0)=0$, then $|f(z)| \leqq |z|$ for $|z|<1$. By omitting the condition $f(0)=0$, Carathéodory and Koebe were able to extend this lemma. If $r<1$, Carathéodory showed that the values of $f(z)$ are in the circle whose center is at the origin and whose radius is $[r+|f(0)|]/[1+r|f(0)|]$, and Koebe showed that the values of $f(z)$ are in a circle whose center is at the point $f(0)$ and whose radius is $r[1+|f(0)|^2]/[1-r|f(0)|]$.

The author obtained a sharper result than the two given above. He showed that the values of $f(z)$ are in the circle whose center is at the point $|f(0)|(1-r^2)/[1-r^2|f(0)|^2]$ and whose radius is $r[1+|f(0)|^2]/[1-r^2|f(0)|^2]$. This circle is entirely contained in the intersection of the above two circles.

3. *A perpetual calendar*, Professor Charles Hatfield, Jr., University of Minnesota.

A simple circular perpetual calendar based upon Gauss' related congruence was presented and demonstrated.

4. *On a continuous function with derivative nowhere*, Professor M. J. Norris, College of St. Thomas.

An example of a function of the type under discussion given by van der Waerden is modified in a way suggested by the use of binary notation rather than decimal notation. The main interest of the paper is in the discussion of differentiability, which can be made almost intuitive.

5. *In defense of the factoring method of solving quadratic equations*, Professor K. W. Wegner, Carleton College.

A defense provoked by the proposal of W. R. Ransom in the *Mathematics Teacher* (March, 1948) that the factoring method be dropped from elementary algebra courses.

6. *A modification of normal curve grading techniques as applied to achievement examinations*, Mr. J. Stanley Hill, Minnesota Mutual Life Insurance Company.

The paper presupposes general acceptance of the belief that normal curve grading techniques have value in certain situations. Frequency distributions of achievement test scores are often markedly skewed in the direction of the lower grades. When usual normal curve grading techniques are applied to such distributions a paucity of A's is produced together with a redundant number of failures. These difficulties are ameliorated by determining separate measures of dispersion for the scores lying above the mean on the one hand and the scores lying below the mean on the other hand. The measures used are identical in definition with the standard deviation except that summation for both numerator and denominator is performed only on variates lying above the mean in one case and on variates lying below the mean in the other case. A "short method" for calculating these measures of dispersion, analogous to the short method for calculating the standard deviation, has been developed. As a practical matter the division between the upper and lower groups for summation is taken at the class limit nearest the mean. The arithmetic work is not in-

creased materially over that required for the usual standard deviation technique; no additional columns on the work sheet are necessary.

7. *Circular nomograms*, Mr. E. C. Varnum, Barber-Colman Company.

Although there are many nomograms in the technical literature, very few are based on the closed, symmetric form of a circle which embodies eye appeal as well as inclusion of all scales in a finite region. During the last year, the author constructed fifteen circular nomograms of which one was published in *Industrial Quality Control* (Jan., 1950), and nine have been printed by Barber-Colman Company for use by its inspection, engineering and sales personnel. Copies of three circular nomograms were distributed to the audience, and the method of constructing the new type of nomogram was outlined.

L. E. BUSH, *Secretary*

THE MAY MEETING OF THE NEBRASKA SECTION

The twenty-sixth annual meeting of the Nebraska Section of the Mathematical Association of America was held at the Nebraska Wesleyan University in Lincoln, on Saturday, May 6, 1950. Professor C. B. Gass presided at the two sessions.

Thirty-eight persons attended the meetings including the following twenty-two members of the Association: M. A. Basoco, H. W. Becker, A. K. Bettinger, Jessie W. Boyce, C. C. Camp, F. Marion Clarke, H. M. Cox, Morris Dansky, H. W. Doss, Jr., J. M. Earl, C. B. Gass, C. F. Gayton, Jr., Edwin Halfar, L. M. Larsen, W. G. Leavitt, W. T. Lenser, E. J. Lowry, C. R. Perisho, H. L. Rice, Lulu L. Runge, R. G. Sanger, and C. J. Tegels.

Officers elected at the meeting were: Chairman, Morris Dansky, Creighton University; Vice-Chairman, C. B. Gass, Nebraska Wesleyan; Secretary, Lulu L. Runge, University of Nebraska. The annual meeting in 1951 is to be held on Saturday, May 5, at the University of Nebraska.

Professor R. G. Sanger of Kansas State College was the guest speaker, and presented a paper at both the morning and afternoon sessions.

The following papers were presented:

1. *Some simple properties of periodic mappings*, by Professor Edwin Halfar, University of Nebraska.

Considered was a class of commutative mappings (f) on an arbitrary point set S into S. A mapping was said to be periodic at a point x if for some n, $f^n(x)=x$ where f^n means n repetitions of f. The principal results were that if $fg=gf$, f periodic at x, $fg(X)\subset X$ for each $X\subset S$, and g is 1-1, then there exist integers p and q such that $f^p(x)=g^q(x)$; if S is a complete lattice ordered by "$\leqq$," then for each p and f monotonic increasing, the set of points $E[f^p(x)=x]$ is a complete lattice ordered by "$\leqq$" and is non-empty.

2. *Multimeter disturbance factors*, by H. W. Becker, Electronic Radio-television Institute, Omaha.

The speaker demonstrated two corollaries of Thevenin's. When an ammeter of internal impedance Z_{AM} is introduced into a circuit of series impedance Z, then the undisturbed current = (measured current) $(1+Z_{AM}/Z)$. When a voltmeter of internal impedance Z_{VM} is put across a branch having impedance Z', in series with a network having impedance Z, then the undisturbed voltage = (measured voltage) $[1+(Z||Z')/Z_{VM}]$, where $Z||Z'=ZZ'/(Z+Z')$.

3. *Comments on elliptic integrals*, by Professor R. G. Sanger, Kansas State College.

In the early part of this century elliptic integrals and elliptic functions were assiduously studied. Since then, courses in elliptic integrals and the associated functions have gradually disappeared from the offerings of departments of mathematics. An attempt was made in this paper to describe elliptic integrals, their properties and associated functions, and to indicate why interest in them has rapidly dwindled.

4. *A locus of navigational interest*, by Professor O. C. Collins, University of Nebraska, introduced by the Secretary.

The locus considered was that of a navigator who sees three fixed markers, and so manoeuvres his vessel that the middle marker remains angularly equidistant from the other two. On a spherical earth, the corresponding locus is that of ships so placed that two fixed stars are sighted at equal bearings to the east and to the west of north. An extension to space navigations was considered.

5. *On the elastic theory of a certain toroid*, by T. I. Gilroy, Creighton University, introduced by the Secretary.

6. *Supervised study laboratory at the University of Nebraska*, by S. R. Barnett, Extension division, University of Nebraska, introduced by the Secretary.

7. *The development of the calculus of variations*, by Professor R. G. Sanger, Kansas State College.

The development of the calculus of variations from the historical standpoint was considered in this paper. The problems which inspired this subject and the results obtained by workers in this field before 1950 were mentioned. Then, the need for and the problem of determining sufficient conditions for problems in this field were considered. Some results which have been obtained in the present century, and the present trends in the development of this subject, were indicated.

8. *Developments of the Nebraska mathematics test*, by Professor H. M. Cox, Bureau of Instructional Research, University of Nebraska.

The speaker reviewed the studies of entrance examinations in mathematics at the University of Nebraska. Beginning in 1939 with a revision of a short algebra test for students registering in the College of Engineering, the program was extended to all entering freshmen and sophomore students before 1942. The 1942 edition of the Nebraska Mathematics Test was planned to integrate high school mathematics, and it provided an array of questions of increasing difficulty for students of various preparations.

The edition of the Test now in use represents a third pattern of measurement, and includes an effort to partially offset the factor of "forgetting." Charts were distributed showing a study of performance of students in mathematics courses in their second semester of enrollment in college.

9. *Use of elementary structures of modern algebra for traditional questions in elementary theory of equations*, by F. Marion Clarke, University of Nebraska.

It was shown how the abstract definitions of group, ring, integral domain, and field can be used by the undergraduate to give logical strength to an analysis of the number and kinds of solutions that exist for an equation in one variable, and for m linear equations in n variables over a given coefficient domain. Examples from the domains of weaker structure give emphasis to the meaning of techniques which have become traditional habits for the domains of real or complex numbers.

10. *Abstract algebra for elementary teachers*, by Professor W. G. Leavitt, University of Nebraska.

This speaker considered the question as to whether or not some knowledge of modern algebra would be useful to teachers of elementary mathematics. A few brief comments were made, first, on the broader aspects of the question, after which were mentioned a number of possible specific applications of a knowledge of abstract theory to elementary instruction.

11. *Note on elementary functions*, by W. T. Lenser, University of Nebraska.

The author noted the parallelism which exists between certain theorems on elementary functions with those on algebraic numbers, even to the details of proof.

LULU L. RUNGE, *Secretary*

CALENDAR OF FUTURE MEETINGS

Joint meeting with American Society for Engineering Education, Michigan State College, East Lansing, June 25–26, 1951.

Thirty-second Summer Meeting, University of Minnesota, Minneapolis, September 3–4, 1951.

Thirty-fifth Annual Meeting, Brown University, Providence, Rhode Island, December 29, 1951.

The following is a list of the Sections of the Association with dates of future meetings so far as they have been reported to the Associate Secretary.

ALLEGHENY MOUNTAIN, Duquesne University, Pittsburgh, Pennsylvania, May, 1951.

ILLINOIS, University of Illinois, Urbana, May 11–12, 1951.

INDIANA, May 5, 1951.

IOWA, Wartburg College, Waverly, April 20–21, 1951.

KANSAS, University of Kansas, Lawrence, April 7, 1951.

KENTUCKY, Eastern Kentucky State College, Richmond, April 28, 1951.

LOUSIANA-MISSISSIPPI, Mississippi State College, State College, February 16–17, 1951.

MARYLAND-DISTRICT OF COLUMBIA-VIRGINIA.

METROPOLITAN NEW YORK, Manhattan College, April 7, 1951.

MICHIGAN, Michigan State College, East Lansing, March 24, 1951.

MINNESOTA, College of St. Benedict, St. Joseph, April 28, 1951.

MISSOURI, Central College, Fayette, April 6, 1951.

NEBRASKA, University of Nebraska, Lincoln, May 5, 1951.

NORTHERN CALIFORNIA

OHIO, Ohio State University, Columbus, April 21, 1951.

OKLAHOMA

PACIFIC NORTHWEST, State College of Washington, Pullman, June 15, 1951.

PHILADELPHIA, University of Pennsylvania, Philadelphia, November 24, 1951.

ROCKY MOUNTAIN, Colorado State College of Education, Greeley, April 20–21, 1951.

SOUTHEASTERN, Vanderbilt University and Peabody College, Nashville, Tennessee, March 16–17, 1951.

SOUTHERN CALIFORNIA, Whittier College, Whittier, March 10, 1951.

SOUTHWESTERN, University of New Mexico, Albuquerque, March 23–24, 1951.

TEXAS, Southern Methodist University, Dallas, Spring, 1951.

UPPER NEW YORK STATE, Hamilton College, Clinton, May 5, 1951.

WISCONSIN, Carroll College, Waukesha, May 12, 1951.

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